DOCUMENT RESUME

ED 213 575 SE 035 601

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TITLE A Simple Introduction to Simulation, Parts I and II.
INSTITUTION Keele Univ. (England). Inst. of Education.
PUB DATE Jul 80
NOTE 187p.; Not available in hard copy due to marginal legibility of original document. For related document see SE 036 394.
AVAILABLE FROM Information Reference Center (ERIC/IRC), The Ohio State Univ., 1200 Chambers Rd., 3rd Floor, Columbus, OH 43212 ($4.50)

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS College Mathematics; *Computer Oriented Programs; Computer Programs; Discovery Learning; Experiential Learning; Higher Education; Instructional Materials; *Mathematical Models; *Mathematics Instruction; Mathematics Materials; *Probability; Problem Solving; Secondary Education; *Secondary School Mathematics; *Simulation; Teaching Guides
IDENTIFIERS *Computer Simulation

ABSTRACT

This collection of documents promotes simulation as an attractive topic in mathematics instruction that provides game-like activities likely to interest pupils, by illustrating the important ideas of probability and showing how mathematical models can be used in practical problems clearly relevant to everyday life. While computer programs written in BASIC are provided, it is recommended that pupils should perform pencil-and-paper simulations in the early stages in order to fully grasp how they work. Part I of the teaching guide has the following chapters: (1) Introductory Examples; (2) Simulation Using Dice, Coins, and Random Digits; (3) Examples of Simulation; (4) Pupil Material: Suggested Worksheets for Younger Pupils; and (5) Computer Simulation. Part II contains: (6) Program Listings, (7) Theoretical Calculations for Some of the Problems, and (8) Markov Chain Processes. The second part also contains four appendices. (MP)

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A SIMPLE INTRODUCTION TO SIMULATION

Part I

F.R. Watson

U.S. DEPARTMENT OF EDUCATION
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July 1980

(i)

Introduction

Simulation is an attractive topic providing game-like activities likely to interest pupils, illustrating the important ideas of probability and modelling, and showing how mathematical models can be used in practical problems clearly relevant to everyday life. It is an excellent illustration of the power of mathematics and of the benefit of taking thought before plunging into action*. (This remark holds good for simulation itself; Appendix IV gives some examples showing how a simulation may be made markedly more efficient by careful design).

This topic provides a fertile field for work with computers and these materials contain listings of computer programs, written in BASIC, which perform many of the simulations. These can be adapted for example, to run on a microprocessor - or may serve as models for the construction of other BASIC programs for simulation.

Although the computer is a powerful tool in simulation work, it is recommended that, in the early stages, pupils should perform pencil-and-paper simulations in order that they may fully grasp the way the simulation works. Even at a later stage, when programming a simulation is the norm, it may be advantageous to check that the program-logic is correct - by comparing with a short pencil and paper simulation, by printing out intermediate quantities, or by doing a dry-run of the program.†

Indeed, if this work is attempted with younger or weaker pupils, it is urged that physical objects should be used to represent (for example) cars in traffic queues, together with coins and dice to produce the random effects. Only when the pencil-and-paper recording is seen as a short cut, and the speed and convenience of random digit tables appreciated, should the 'concrete' approach be dropped with these pupils. If the plunge into abstraction is too sudden, there is the danger that the whole investigation will become a meaningless ritual, devoid of connection with the real world.

* Ormell in the paper 'The Electronic Bus Stop' gives an excellent example. Simulation of a proposal for operating a minibus service in Adelaide indicated that it would not be profitable; a businessman who thought he knew better abandoned the project after one day's operation! (Mathematics Applicable Group, School of Education, University of Reading, July 1979).

† As an example, an early version of the program FIV6 contained an error (Line 230 read IF K * RND < 1 THEN A = A+1: GOTO 270). (See p 76) The resulting output aroused suspicion and the error was located after intermediate values of A & B had been printed out. (What would be the effect of this error? See over, p4).

Though the material is likely to be of most use at about sixth-form level, many of the ideas could be used, in suitably simplified form, by teachers with much younger pupils. Here simulations are perhaps best treated rather like the familiar board games; computer results may be quoted, but there is a danger that numerical output may fail to be related to the underlying situation which is being modelled, and so work with dice and coins may be preferable here. Section 4 gives some suggestions for pupil work-sheets, which are also available separately, with full-size "templates" for playing some of the games.*

In another context, some of the material may be useful as an informal introductory treatment for first year undergraduates in mathematics or computer science, who have had no prior acquaintance with the topic.

A number of examples have been taken from other sources; these are acknowledged, and some references are given to other material. I am grateful to Dr. P.W. Jones and Professor C.M. Reeves who kindly read through this material - though they are in no way responsible for any errors or inadequacies! I should also like to thank Miss Annette Brindley for typing the drafts and Mrs. C. Lea for printing this booklet.

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* "Some exercises in simulation: pupil materials." Institute of Education, University of Keele, 1980.

† See p (i). A second random number would be chosen, so that prob (K* RND < 1) is $\frac{1}{6}$ instead of $\frac{1}{2}$; this gives far more sixes than fives.

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Section 1
Introductory Examples

We consider first, five simple problems, using them to illustrate the idea of simulation.

(A) Rumour Problem. A rumour spreads among a group of 6 friends as follows. When one person hears the rumour he telephones one of the other 5, chosen at random, to pass it on, and having done so takes no further interest in the rumour. The second person likewise passes on the rumour to just one other, chosen randomly; if the message is passed to someone who has already heard it, both individuals stop transmitting it, and no further people are informed.

Is it likely that all 6 will get to hear of the rumour? On average, how many of the 6 will get to hear of it? (See p 104)

(B) Road-up. Temporary traffic lights are installed to control one-way traffic when a section of a road running N-S is under repair. Because vehicles cannot clear the one-way section instantaneously, there will be a lag between the halting of traffic in one direction, and the release, by a green signal, of traffic in the other. The rate of flow of traffic in the two directions may differ (e.g. at rush hour). How should the timing sequence for operation of the traffic lights be adjusted for best results? (See p 7)

(C) Christmas Pudding. Twelve people sit down to Christmas dinner and are served a Christmas pudding containing 12 coins. When the portions are given out, 5 people find there is no coin concealed in their portion, and one person has 3 coins.

Has the pudding been fairly divided? How many portions can be expected to contain 0,1,2,3.....coins?

How many coins should be put in the pudding so as to make it very unlikely that anyone is 'left out'? (See p 107)

(D) Collector's Problem. A cereal manufacturer provides a small plastic badge with each packet of cereal. If there are 6 different types of badge and they are randomly distributed one to each packet, how many packets of cereal must I buy, on average, to obtain a complete set of the badges? (See p 107)

(E) Two-Stage Reaction

A certain substance A is transformed by a chemical reaction into a substance B, which in turn is transformed into a substance C. The rate at which a substance is transformed is proportional to the amount of it which is present. If there are a gm. of A present initially (and none of B and C) find how the amounts of A,B,C depend on time, the maximum amount of B which is ever present, etc. (See p 118)

Discussion

These problems have been chosen to introduce the idea of simulation; all involve fairly simple situations, and all except B can be analysed mathematically without too much difficulty. (See below, pp 104). Each of these small problems, though presented in a rather limited context, also mirrors other related situations of great practical importance. Thus the spread of rumours is analogous to the spread of epidemic diseases, and the problem of spraying paint-droplets randomly so as completely to cover a surface resembles those of C and D. Problem B can clearly be extended to more complex forms of 'traffic' whether of cars along a road, or telephone calls to an exchange, or people waiting for service in a supermarket, while E exemplifies a large class of problems involving chemical reactions, radioactive decay etc. These more realistic problems, and others we shall mention later, are often extremely complex mathematically, and some indeed are quite intractable in terms of obtaining an explicit mathematical solution. Such problems may be approached by the method of simulation, which will be described and illustrated in what follows.

You may have noticed that several of the five problems, as given above, are incompletely formulated. Thus, in the last sentence of problem C the meaning of 'very unlikely' must be specified before the question can be answered - we might ask "How many coins are needed if the probability that no-one is 'left out' is to be at least 0.95?" or, more comprehensively, seek the relationship which tells us how many coins are needed for any specified level of probability that no-one is 'left out'.

There is a more fundamental omission in problem (B), where 'best results' may have several different definitions, possibly mutually contradictory - clearly if the lights are set permanently on green for N-bound traffic this will please some motorists and not others! 'Best results' might be interpreted as 'producing minimum average waiting time per vehicle', or 'ensuring that no vehicle is delayed for more than T minutes' or 'ensuring that the queue at

either end never exceeds N vehicles' - in general, these will lead to different suggested timing sequences.

One characteristic which all four problems A - D have in common is that they involve the operation of chance factors. We describe such problems as stochastic, and simulation methods are particularly useful in such situations. However, simulations may also be used to study non-chance (deterministic) situations such as that in E, and other examples of these will be given later. Linked chemical reactions like the one described in E are quite common in industry, and the method of simulation - in this case essentially a step-by-step method of solving the associated set of differential equations - is useful in examining complex systems of reactions, estimating reaction speeds, product yields, etc. Problem E is also typical of radioactive decay processes - for example, the chain of transformations by which uranium is spontaneously converted, ultimately into lead.

Other deterministic situations are illustrated in problems involving the motion of a projectile allowing for air resistance, (p 97), pursuit problems (p 94), predator-prey interactions (p 95), and the spread of rumours and epidemics (p 94 , and c.f. p 65).

The use of simulation

Setting up a simulation can help us by giving greater understanding and by improving efficiency. The process of constructing a model of a real situation increases insight - we need to think what aspects of the situation are important and how they interact. Running the simulation may serve to show the relative importance of the features involved or may indicate, if the simulation does not correspond to reality, that significant aspects of the situation have been omitted.

Secondly, if the model of reality is a good one, i.e. if it behaves in the same way as the real situation does, we can use it to predict the effect of proposed changes - altering the traffic light setting in (B) above, introducing an appointments system in a doctor's surgery (p.19), increasing the number of serving points in a supermarket (see Appendix I, p.159), etc. Industry and Commerce make use of such 'theoretical' simulations as well as 'physical' simulations like wind-tunnel experiments and chemical pilot plants. To alter some aspects of the model and observe the resulting change is usually easier than to manipulate the real situation, it may also be cheaper or less dangerous - consider such examples as learning about company management or investment thro' business games, training pilots on a 'simulator', studying a nuclear reactor accident...

Section 2

We now illustrate the idea of simulation by considering the problems A,B,C,D described earlier. In a simulation we attempt to 'model' the process or situation we are interested in, and perform an experiment - the behaviour of the model then tells us something about the situation, and it is usually easier to change details in the model than to alter corresponding features of the real situation.

(a) For a model of the rumour problem (A) we shall use 6 cards labelled 1 - 6 and an ordinary die.* We start by marking card '1' with a cross to indicate that person 1 has heard the rumour. We roll the die to determine who is next informed of it. Suppose the die shows '3' - then card '3' is marked with a cross, to indicate that no. 3 knows of the rumour and the die is rolled again. (If a number is obtained twice in succession - ie. corresponding to the card just marked - this throw is ignored). When the die indicates a card which is already marked ("person already knows rumour") the experiment stops, and we note how many cards are now marked ("how many persons have heard the rumour"). The experiment may now be repeated many times - the pattern of results will tell us how we may expect the rumour to be propagated in the situation we are modelling. In a stochastic problem of this type it is necessary to repeat the simulation several times, since each occasion will produce slightly different results - however, an advantage of simulation is that it is easy enough to repeat the experiment, and also to vary the conditions in a way which may not be at all easy in the original problem.

If 12 people are involved, we can use two dice - if the first displays an even number we use the score on the second die, if it displays an odd number we add 6 to the score on the second die. In this way, each of the numbers 1 - 12 appears with equal probability. We can change the model by allowing each person to pass on the rumour to 3 others (in an epidemic study this would be analogous to 'remain infective for three days'), stop retailing the rumour when he has heard it <u>twice</u> (which might be a more realistic condition), and so on.

* Alternatively we could draw tickets labelled 1 - 6 out of a hat - cf. Bingo or Tombola. The principle is the same, though use of a die may be more convenient. (The 'Bingo' method can be extended to more general probability distributions in which the outcomes are not all equally likely, by having several labels bearing the same number. See also pp.162 and 175.)

An example of a simulation for the first (simple) version of the problem is:

Dice score	1	6	~~4~~	5	~~3~~	2	1	3	~~3~~	2	5	3	5	~~5~~	2	6	2
Number hearing rumour	1	2	2	3	4	5		1	1	2	3		1	1	2	3	

Average number told = $\frac{5 + 3 + 3}{3}$ = 3.7 (See p 106).

Clearly dice could be used to simulate problem (D), which is equivalent to "How many rolls of a die do I need to obtain each of the scores 1 - 6 at least once?", and problem (C) ("Twelve numbers are chosen at random, with replacement, from the set {1,2,.....12} ; how many numbers are chosen 0,1,2,3,....times?")

Examples

(D) Die shows 2 6 1 1 3 5 2 5 2 6 1 5 4 | 5 6 2.....

13 throws, corresponding to "13 packets of cereal to obtain a complete set" (See p 107).

(C) Dice show	odd	even	odd	odd	even	even	odd	even	even	even	even	odd
	2	5	5	2	4	6	5	1	5	6	5	3
corresponding to	8	5	11	8	4	6	11	1	5	6	5	9

Number from {1,2,...12}	1	2	3	4	5	6	7	8	9	10	11	12
No. of times selected	1	0	0	1	3	2	0	2	1	0	2	0

No. of times selected	0	1	2	3
Frequency	5	3	3	1

Total = 12 (See p 107)

The simulation could be continued each time until <u>every</u> number had been chosen at least once - this would estimate how many coins need be placed in the pudding so that every portion contains at least one.
(Note that this is now the <u>same</u> problem as D).

(b) Random Numbers

A sequence of random digits like that given below is useful in simulation work. (In point of fact, most simulations are carried out using a computer,

which also generates the random numbers required (See p14, 61)
However, for the moment we shall perform simulations using 'pencil-and-paper' methods to illustrate the principles).

Random digits:-

73875 39496 06385 48458 40[illegible] 74383 22814 36752 10606 48774

These may be regarded as 50 digits randomly selected from the set $\{0,1,2,.....9\}$ or, by inserting a decimal point before each group of 5 digits, (e.g. 0.73875), as 10 randomly chosen numbers from the (continuous) interval $0 \leq x < 1$

(Since only 5 dec.pl. are given, the sampling is not, strictly speaking, from the set of all possible numbers in the interval (e.g. the value 0.738753 could not arise) but for practical purposes it may usually be considered to be so).

We can use random numbers instead of dice and coins to produce random choices:-

e.g. the sequence 7 3 8 7 5 3 9 4 9 6

could be used to'mimic' the tossing of a coin

T T H T T T T H T H

(using T = odd, H = even digit),

and 0 6 3 8 5 4 8 4 5 8

could mimic a die

6 3 5 4 4 5 ..... (ignoring 0 and 7,8,9).

Example

(C) again: we wish to choose digits at random from $\{1,2,3,.....12\}$

Consider the random digits:-

7 3 8 7 5 3 9 4 9 6 0 6 3 8 5 4 8 4 5 8

and take them in pairs, considering each pair as a two digit number. Ignore occurrences of 97,98,99,00, and, in every other case, take the remainder on dividing by 12 (i.e. reduce the two digit number , modulo 12)

We obtain	73	87	53	94	96	06	38	54	84	58	40	33
corresponding to the choices	1	3	5	10	0 (ie.12)	6	2	6	0 (ie.12)	10	4	9

Number from $\{1,2,....12\}$	1	2	3	4	5	6	7	8	9	10	11	12
No. of times selected	1	1	1	1	1	2	0	0	1	2	0	2

No. of times selected	0	1	2	3
Frequency	3	6	3	0

Total = 12 (See p 107).

(Note: the 2 digit numbers CC, 97, 98, 99 are ignored in order that the probability of occurrence of each remainder 0,1,2,.....12 should be the same ($= \frac{1}{12}$). If they are included, the remainders 0,1,2,3 will be slightly more likely to occur than the others).

Exercise

Use the table of random numbers (p 14) to simulate the following situations:-

(i) (c.f. problem D)

(a) Collecting a set of 10 cards from packets of tea; (use the random digits 1 - 9 and 0).

(b) Collecting a set of 20 cards. (Use pairs of random digits - consider the remainder, mod 20) (See p 107).

Perform several simulations, and in each case observe the average value of the quantity (no. of packets etc.) which is of interest, and also its variability between simulations.

(ii) (c.f. problem A). Investigate the spread of a rumour in a group of 10 people. (See p 106)

Example (c.f. problem B), p 1.

Suppose the time lag between green signals (to allow traffic to clear the one way section) is 5 seconds, and that just two vehicles can pass through the section in 5 seconds. Suppose also that N-bound traffic arrives at the average rate of 6 vehicles per minute (½ every 5 sec), and traffic travelling S at 12 per minute (1 every 5 seconds).

Four traffic light patterns are under consideration, as shown. Each consists of 4 phases:-

Phase			Pattern:	(a)	(b)	(c)	(d)
I	N-bound lights green			5 sec	10	25	15
II	Both lights red			5 sec	5	5	5
III	S-bound lights green	for		5 sec	10	25	30
IV	Both lights red			5 sec	5	5	5

We shall illustrate the simulation for patterns (a) and (b) using 5 sec, as a unit of time, and tossing coins to obtain probabilities of arrivals. For N-bound traffic, 'coin shows H' corresponds to arrival of vehicle ($p = \frac{1}{2}$ corresponding to $\frac{1}{2}$ vehicle in 5 sec, 6 vehicles per minute)

For S-bound traffic, 2 coins are tossed: HH ⟶ 2 vehicles in unit time interval (5 sec)

HT, TH ⟶ 1 vehicle
TT ⟶ 0 vehicles
} in unit time interval

Pattern (a): Explanation (see table below)

Time period (1) One vehicle arrives, S-bound, and can pass through. No queues formed.

Time period (2) One vehicle arrives from each direction. Two queues formed of size 1.

Time period (3) One vehicle from each direction. 2 pass through N-bound. S-bound queue is 2.

Time period (4) One vehicle arrives S-bound. No vehicles pass. Queues are 0 and 3.

Time period (5) One vehicle arrives from each direction. Two vehicles pass S-bound; queues are 1 and 2.

Time period (6) One N-bound, 2 S-bound vehicles arrive; none move off. Queues are 2 and 4, etc.

('Queue' refers to queue at end of time period.)

Time period		1	2	3	4	5	6	7	8	9	10	11	12
Coins for N		T	H	H	T	H	H	T	H	H	T	H	H
Coins for S		TH	TH	TH	TH	HT	HH	HT	HH	HT	HH	HH	HT
N bound Queue	(0)	0	1	0	0	1	2	0	1	2	2	1	2
Arrival		0	1	1	0	1	1	0	1	1	0	1	1
Departure		0	0	2	0	0	0	2	0	0	0	2	0
Phase		I	II	III	IV	I	II	III	IV	I	II	III	IV
S bound Queue	(0)	0	1	2	3	2	4	5	7	6	8	10	11
Arrival		1	1	1	1	1	2	1	2	1	2	2	1
Departure		1	0	0	0	2	0	0	0	2	0	0	0

Comment

We note the rapid build up of queues, especially for S-bound traffic, which arrives with twice the density of N-bound traffic. By chance, arrivals are also unexpectedly heavy in this period - (8N, 16S instead of 5N, 10S). This traffic-light pattern would be useful for very light traffic densities since "waiting for the lights to change" is reduced to a minimum, but is inefficient in these circumstances of heavier traffic, since both lights are at 'red' for 50% of the time.

Pattern (b)												
Time period	1	2	3	4	5	6	7	8	9	10	11	12
Coin for N	T	H	H	T	H	H	T	H	H	T	H	H
Coin for S	TH	TH	TH	TH	HT	HH	HT	HH	HT	HH	HH	HT
N bound Queue (0)	0	1	2	0	0	1	1	2	3	1	0	1
Arrival	0	1	1	0	1	1	0	1	1	0	1	1
Departure	0	0	0	2	1	0	0	0	0	2	2	0
Phase	I_S	I_S	II	III_N	III_N	IV	I_S	I_S	II	III_N	III_N	IV
S bound Queue (0)	0	0	1	2	3	5	4	4	5	7	9	10
Arrival	1	1	1	1	1	2	1	2	1	2	2	1
Departure	1	1	0	0	0	0	2	2	0	0	0	0

Note: We use the same arrival pattern as in the preceding example, for comparison. The rows for arrivals may be completed first, then the zeros inserted in the departure rows, using the 'Phase' row as a guide. (I_s means 'S traffic free to move during this period'). Finally departures and queue rows are completed successively:

existing queue + new arrivals - possible departures = new queue at beginning of next period.

This pattern gives slightly smaller queues at the end of 12 time intervals (1 minute), although whether this is a genuine improvement would need to be checked by running the simulation for a longer period.

Exercise

(i) Extend the simulation of (a) for a further 12 time intervals, using coins to obtain the random vehicle arrivals

(ii) Do the same for pattern (b), but instead of coins use single digits from

the random number tables (p 14) as follows:-

For N-bound traffic 1 - 5:- vehicle arrives; 6-9,0:- no vehicle.

For S-bound traffic; 1 or 2:- no vehicle; 3,4,5,6:- one vehicle; 7,8:- two vehicles; 9,0:- ignore.

(iii) Simulate pattern (c) over 24 time intervals (2 minutes). (The 'Phase' row will now read I I I I I II III III III III III IV I I I I I .....etc).

(iv) Do likewise for pattern (d).

Note: We should expect an improvement with (c), (since less time is wasted with both lights at 'red') and a further improvement with (d), where the 'green' periods are now better matched to the rates of traffic flow. Several extended simulations were carried out by computer to investigate these patterns (see p 79, ABOVE), with the following results

No. of cycles simulated = 100

Time periods			4sec	4sec	12sec	11sec
'Phases'			33⅓	50	100	83⅓
		Pattern ---→	a	b	c	d
Queue lengths during simulation	N bound	average	1·84	0·99	1·69	3·10
		maximum	7	4	6	11
	S bound	average	101·7	50·7	110·7	6·09
		maximum	206	193	210	75

Note the dramatic improvement for S bound cars in pattern (d). In (c) the expected number of arrivals (12 periods) is 12; maximum capacity of road is 5 × 2=10; corresponding figures in (d) (11 periods) are 11 and 12.

Simulation of Dice and Coin Experiments Using Random Numbers

Example (1)

Toss a penny until a sequence of 3 consecutive heads (i.e. HHH) appears. Find the average number of tosses needed.

Use single random digits (e.g. even = H, odd = T).

40338 42477 78804 36272 72043 07958 67158 60979 79891 92409

HHTTH HHHTT THHHT THHTH THHHT HTTHH HTTHH HHTTT TTHTT THHHT

7 3 10 13 12

Average = $\frac{49}{5}$ = 9.8 (Theoretical average = 14 - see p 109)

Example (2)

Two dice are rolled. What is the probability of each of the scores 2 - 12?

(Use single digits, (disregarding 0,7,8,9), in pairs, to represent the two dice).

e.g. 40338 42477 78804 36272 72043 07958 67158 60979 79891 92409

4 33 42 4 36 2 2 43 5 6 15 6 1 24

Scores: 7 7 6 7 8 4 7 11 6 7 6 14

The theoretical probabilities are

S=2	3	4	5	6	7	8	9	10	11	12
$p=\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(see p 109)

Computer simulation of 360 throws produced the following results:-

(two experiments)

Score		2	3	4	5	6	7	8	9	10	11	12
Frequency	(a)	9	25	34	41	51	48	48	42	26	27	9
	(b)	12	12	25	52	55	44	52	33	41	25	9

Example (3)

Repeat question 2, (i) for three dice, (ii) for four dice.

Theoretical probabilities are calculated on pages 109-10.

Computer simulation of 2160 throws and 1296 throws respectively produced the following results:-

3 dice (two experiments)

Score		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Frequency expected		10	30	60	100	150	210	240	270	270	252	210	150	100	60	30	10
Frequency observed	(a)	7	24	50	94	129	222	259	262	276	249	232	144	94	47	25	12
	(b)	10	33	67	106	116	208	225	260	252	245	210	157	95	66	30	5

4 dice (three experiments, results cumulated)

Score	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
Frequency expected	1	4	10	20	35	56	80	104	125	140	146	140	125	104	80	56	35	20	10	4	1
Frequency observed (1)	2	3	12	26	49	57	75	94	111	116	123	127	116	116	77	65	37	22	8	4	1
(1)&(2)	5	8	20	46	66	116	152	187	250	285	267	273	243	210	163	104	65	40	16	8	1
(1)&(2)&(3)	5	11	29	67	101	171	240	291	367	408	449	432	357	285	203	140	95	55	24	12	2

Example (4)

(i) Three coins are tossed. What are the probabilities of each of the outcomes T^3, TH^2, HT^2, H^3, ?

(ii) Repeat for four coins and for five coins.

The theoretical results follow the Binomial distribution:-

(3 coins) $p(T^3) = (\frac{1}{2})^3$, $p(HT^2) = 3.(\frac{1}{2}).(\frac{1}{2})^2$, $p(H^2T) = 3.(\frac{1}{2})^2.(\frac{1}{2})$, $p(H^3) = (\frac{1}{2})^3$

and for four coins p^4, $4p^3q$, $6p^2q^2$, $4pq^3$, q^4 } where in each

and five coins p^5, $5p^4q$, $10p^3q^2$, $10p^2q^3$, $5pq^4$, q^5 } case $p = q = \frac{1}{2}$

Simulation for five coins:- (even = H, odd = T).

49487 52802 28667 62058 78228 47041 18319 17889 45869 14454

H^3 H^4 H^4 H^4 H^4 H^3 H H^2 H^3 H^3

Observations	H^5	H^4	H^3	H^2	H^1	H^0
Frequencies	0	4	4	1	1	0

Total 10

Repeat for 100 (or 96) groups of 5 digits. The expected frequencies are:-

H^5 H^4 H^3 H^2 H^1 H^0

3 15 30 30 15 3 Total = 96 (see p 111).

Example (5)

A coin is tossed repeatedly. If four heads appear successively, we call this a 'run' of length 4. Thus the sequence

HHTTTHTTTHTHHH contains { 1 run of length 5; 2 runs of length 3; 1 run of length 2; 3 runs of length 1

Find the average length when a coin is tossed 100 times. (Use random digits, even = H, odd = T).

(Equivalently, find the number of runs in 100 tosses; $\frac{100}{\text{Av. length}}$). (See p 111)

(Engel (1977) gives the following example of a sequence containing 23 Hs and 27 Ts, which looks at first sight as though it might have been obtained by repeatedly tossing a coin:-

HTHTHHHTHTTHTHHTTTHTHHTTHTHTHHTTTHTHTHTHHTTHTHTHTT. It contains 36 runs, an unusually large number, which would be very unlikely to arise in such circumstances; the expected number is 25. One possible way in which such a sequence might be generated is by a system in which successive trials are not independent - in this case, the system behaves like a 'flip-flop' device, tending to change states - the probability of H following T is higher than that of H following H, and vice versa). See p 111 + p 86 Runs.

To conclude this section we discuss problem E, as an example of a deterministic simulation (see also p 42 where a similar problem is treated as a chance phenomenon - i.e. stochastically). Suppose that initially 100g. of A is present and that A is transformed into B at the rate of 0.2g per second per g of A present. (i.e. that $\frac{da}{dt} = -0.2a$, where a is the amount in g of A). Suppose that the rate of transformation of B is 0.3g per g of B, per second, i.e. $\frac{db}{dt} = -0.3b + 0.2a$. (The second term appears since A is continually being transformed into B). Finally $\frac{dc}{dt} = +0.3b$. As a first approximation we can consider the process for 1 second and assume that the rates of change are constant, i.e. that 20g of A will change into B during the first second. (This will be an overestimate, since as soon as some of A is transformed, the speed of reaction will begin to decrease slightly). Continuing with 1 second steps we obtain the following table.

Time (period)	Amount of A a	Change in A a	Amount of B b	Change in B (c) C	Amount of C c
0	100		0		0
0-1		-20		0	
1	80		20		0
1-2		-16		+6	
2	64		30		6
2-3		-12.8		+9	
3	51.2		33.8		15
3-4		-10.2		+10.1	
4	41		33.9		25.1
4-5		-8.2		+10.2	
5	32.8		31.9		35.3
5-6		-6.6		+9.6	
6	26.2		28.9		44.9

A computer simulation using steps of $\frac{1}{10}$ sec, produced a more accurate result:-

6	29.8		27.3		42.9

(see p. 93, DECAY2).

(The theoretical predictions are :-

$a = 100e^{-0.2 \times 6}$ 30.12

$b = 100 (e^{-0.2} - e^{-0.3})$ 27.18

see p 118) 42.70

RANDOM DIGITS

49487	52802	28667	62058	78228	47041	18519	17889	45869	14454
48029	91539	46317	84803	86056	62812	33584	70391	77749	64906
25252	97738	23901	11106	86864	45808	22557	23214	15202	54268
02431	42193	96960	19620	29188	05863	92900	06836	13433	21709
12146	18256	81958	44757	94641	36145	68164	20514	64246	30707
38354	94318	89709	3690	11279	09918	39783	72845	05688	11066
83215	01923	40332	41510	88757	94672	33960	45001	25159	44843
05322	65289	61169	46185	73169	46124	65540	92450	56250	11256
07029	77386	73315	64435	54358	94745	43115	89410	76306	38275
84399	49933	41614	01727	15018	89752	19021	78851	56421	52618
77307	63397	24658	82281	85193	49115	39380	22491	11438	29755
19336	16998	24792	99066	83276	09552	94006	50800	49951	43903
87830	78705	38110	63800	75012	76532	66504	12994	24304	38031
53882	35236	04456	56952	18994	74274	84216	27045	80591	73859
32336	42059	57422	77740	17566	95746	68237	29669	08655	81854
31787	73395	74353	94135	66821	52667	83411	26337	92090	11237
04578	72211	26343	92853	06604	25507	88330	41266	58240	79974
96802	00228	28235	29364	70508	13483	85339	67426	28198	24786
98303	87909	88623	77887	35876	84564	70532	16534	66809	51141
92676	84479	59851	81390	73804	25470	83752	69061	32666	83258
07263	07904	88013	01593	99133	91669	58594	24225	28113	14105
63159	94904	62952	68964	20459	57379	72400	49994	49243	55402
25208	50946	68262	32721	90125	65570	96265	33087	35901	87616
52002	00250	31287	10834	54224	77960	45032	28973	21680	09967
45886	35785	73120	40021	02600	25018	27295	11871	83923	90424
02954	69269	58606	25751	18848	55963	95398	24750	93726	15704
63049	81171	46338	92242	30310	88763	95435	29327	65930	41278
59766	70709	18660	32465	58081	60138	17249	56073	09131	41364
70447	05853	31665	58136	67004	75555	44434	54205	75671	58929
66187	73322	65198	49725	15674	59235	04333	41693	11646	55695
61829	28583	72852	06451	06433	04144	18042	55255	06897	62128
65967	45856	31970	96283	35376	22003	50403	00360	45020	27448
30945	68109	13647	05939	42346	93280	60010	01227	53357	69623
02856	57062	32727	90887	60938	17194	49207	50824	53003	25372
71545	43182	97803	25348	68494	61713	14087	60870	08801	00165
20605	75090	61218	52289	36108	13556	94495	11835	79346	18219
77380	72552	69067	33429	78625	28192	24023	02936	66980	72504
62964	70490	11194	99427	04639	79840	80017	02142	67798	74738
42200	74969	71191	98932	66492	11469	33569	96173	21643	05389
73682	10211	76404	50482	10278	82706	99524	40497	62109	63678
59753	69183	47925	90607	25842	30292	86475	09332	66541	17572
96509	63605	50598	24774	96777	97174	46765	45648	06030	53790
2 792	73944	43018	77203	50427	03412	26489	11163	95422	27802
7 195	99420	27527	40863	07886	85724	15491	36346	43311	13824
2 064	08002	00220	27472	33997	49579	97363	70416	02039	54828
3491	86407	00940	17499	87354	19196	99451	31342	17700	12531
6321	90106	63281	10162	70300	87543	42847	55841	80139	17401
75146	93317	64587	73431	78931	66339	92395	49384	72949	18658
32312	39008	75952	94025	53088	36053	06689	36188	23474	34271
83911	88898	12219	27411	26367	95905	88074	09222	52808	00958
19788	73456	81982	47809	76111	13861	32642	80206	25793	24188
23535	41901	37585	98187	73413	76642	80237	29608	01025	28180
22498	12201	25122	40265	33118	39716	64453	56647	80847	05902
37769	21075	34436	04510	63818	77301	62634	29291	61353	69073
34192	73993	49121	40143	17859	32367	45874	34259	82385	98163
70361	95172	96521	65131	41333	66632	29016	27020	77539	92389
48621	77582	97729	16193	24084	10565	20654	81793	24158	19720
65063	32941	17590	98798	49707	13385	73132	41547	93335	56876
59534	41718	14697	37164	45544	93060	32544	67999	99915	85125
65625	03131	91394	24261	32690	86310	88733	91620	52490	61283
60681	85150	43774	71808	75989	98602	25293	61627	03406	25726
15796	74493	11682	60272	34033	54156	69568	95990	98755	44368

Note: Ways of Producing Random Digits

Perhaps the simplest way of producing a set of random digits from the set $\{1, 2, 3, 4, 5, 6\}$ is by rolling a die. Similarly a coin can be tossed to produce random sequences of '0's' or '1's' (eg 0='head', 1='tail').
As an alternative to a cubical die, random digits from the set $\{1, 2, 3, 4, 5, 6\}$ may be obtained by labelling the 6 faces of a hexagonal pencil or Ballpen.
Tables of random numbers, like that on p 14 are also available, whilst for computer simulations the random number generator on the computer itself may be used. This note suggests ways in which you can generate your own random digit sequences for different purposes using

(i) a modified cubical die

or (ii),(iii) a calculator.

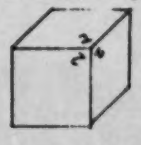
(i) Cubical die

(a) A cube has 8 vertices, which may be labelled 1 - 8. Using the rule 'select the uppermost vertex nearest the eye; in the case of a tie choose the R.H. vertex', random digits may be selected from the set $\{1, 2, \ldots\ldots 8\}$.
In a board game, arguments might arise about the correct choice of vertex (in which case an umpire might be needed).
In a simulation the choice is not important (provided that the experimenter does not deliberately choose an outcome to provide a 'freak' result).

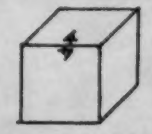
(b) Similarly the 12 edges may be labelled, allowing choice of random digits from $\{1, \ldots\ldots\ldots 12\}$ with the rule - "take the uppermost edge nearest the eye".

(c) The 24 face corners may be labelled with the rule - "take the vertical face nearest the eye, and the number in top right hand corner of the face". (Selection from $\{1, \ldots\ldots 24\}$).

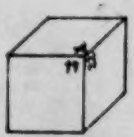
(Since there are just 24 rotational symmetries of the cube, this is the largest set of random digits obtainable in this way).



(a) Vertex-labelled cube, displaying '2'

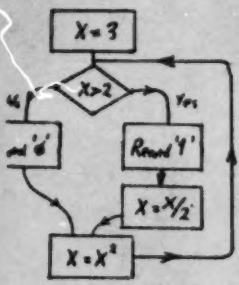


(b) Edge-labelled cube, displaying '4'



(c) Face-corner labelled cube, displaying '19'

(11) Random Sequence from $\{0,1\}$ using a calculator



(i) enter 3 as X
(ii) if X>2, record '1' and divide by 2 else record '0'
(iii) square X
(iv) goto step (ii)

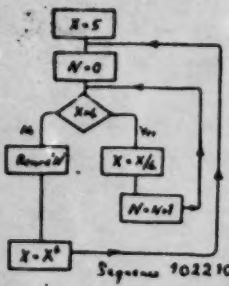
The sequence is 'pseudo-random' and in effect gives the binary value of x where $2^x = 3$
(Consider $3 = 2^x = 2^{1+y}$.
Then $2^y = \frac{3}{2}$ so that
$2^{2y} = (\frac{3}{2})^2 = 2.25 = 2^{1+z}$,
where $z = \frac{2.25}{2}$)

Sequence is 1100101011....

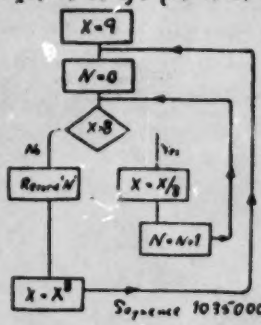
Since x is known to be <u>irrational</u>, the sequence will <u>recur</u> only because of the finite accuracy of the calculator used. (eg. it may work to only 8 decimal digits).

In practice, it is found that this <u>pseudo-random</u> sequence contains '0's' and '1's' in approximately equal proportions, and in other ways behaves as a genuinely random sequence.

<u>Modifications</u> will produce random sequences from $\{0, 1, 2, 3\}$, $\{0, 1, 2, 3, 4, 5, 6, 7$



Sequence 1022103....



Sequence 1035000271....

(iii). The number π is <u>transcendental</u> (i.e. is not the root of an algebraic equation with integer coefficients). If we disregard the finite accuracy of the calculator, we may obtain a <u>non-periodic</u> sequence

by INT $((\pi - 3) \times 10^5)$ 14159
INT $((\pi^2 - \text{INT}(\pi^2)) \times 10^5)$ 86960
INT $((\pi^3 - \text{INT}(\pi^3)) \times 10^5)$ 00627
. 40909
01968
(i.e. we take the 1st 5 dec. places of π, π^2, π^3, . . . etc.) 38919
(When the displayed value becomes large, 5 dec. places cannot be obtained. At this stage we may subtract the integer part before multiplying by π, and continue). 22268
84118

This gives sets of 3 digits at a time, which have been found to be suitable for use as pseudo-random numbers, although the approximation used for π is clearly a <u>rational</u> number.

Section 3

Further Examples of Simulation Exercises Using Random Numbers

Example I (Adapted from Selkirk, K (1974))

Patients arrive at a doctor's waiting room in the period 9-10am. Suppose 12 patients arrive in the hour, an average of one every 5 min. In practice, they will not arrive at 5 min. intervals but randomly.

(a) We can simulate this process by choosing 12 two-digit random numbers in the range 01 to 60 - these give the arrival times. Then if each patient is seen for exactly 5 min., we can follow through the process to examine how much time is wasted by the doctor and by the patients.

Example

Time of Arrival (random)	See Doctor	Leave	Wait
9.10	9.10	9.15	0
9.12	9.15	9.20	3
9.18	9.20	9.25	2
9.24	9.25	9.30	1
9.28	9.30	9.35	2
9.29	9.35	9.40	6
9.32	9.40	9.45	8
9.36	9.45	9.50	9
9.37	9.50	9.55	13
9.41	9.55	10.00	14
9.49	10.00	10.5	11
9.57	10.05	10.10	8
			Total = 78

Doctor wastes 10 min

Average = 6.5 mins

Do this simulation yourself - your results will differ somewhat from those above

(b) In practice the time spent with the doctor will be random - we might assume it to be chosen randomly from the range 1 - 9 mins. (with an average of 5).

Example (using the arrival times on previous page)

Time of Arrival (as previously)	See Doctor	Random no.	Leave	Wait	Doctor
9.10	9.10	5	9.15	0	10
9.12	9.15	3	9.18	3	
9.18	9.18	3	9.21	0	3
9.24	9.24	4	9.28	0	
9.28	9.28	1	9.29	0	
9.29	9.29	4	9.33	0	
9.32	9.33	5	9.38	1	
9.36	9.38	6	9.44	2	
9.37	9.44	7	9.51	7	
9.41	9.51	6	9.57	10	
9.49	9.57	8	10.05	8	
9.57	10.05	2	10.07	8	
		54		39	
	Average	$\frac{54}{12} = 4$		Average = 3.33 min	

Doctor wastes 13 min ; finishes a little earlier.

Average wait is reduced - but probably due to chance happening that average stay with the doctor is only 4.5 min.

This illustrates that a single run of the simulation may not give a typical result. (We are, after all, investigating a situation dependent on chance). In practice several runs should be done and the results compared.

Do this simulation yourself.

(c) We could investigate the effect of an appointments system; say bookings at 5 min. intervals with patients arriving randomly 0 - 8 min early.

Example

Booked Time	R.no.(0-8)	Arrival Time	See Doctor	R.no.(1-9)	Leave	Wait	Doctor
9.00	7	8.53	9.00	9	9.09	7	
9.05	8	8.57	9.09	1	9.10	12	
9.10	4	9.06	9.10	3	9.13	4	
9.15	8	9.07	9.13	4	9.17	6	
9.20	6	9.14	9.17	7	9.24	3	
9.25	0	9.25	9.25	4	9.29	0	1
9.30	4	9.26	9.29	2	9.31	3	
9.35	7	9.28	9.31	5	9.36	3	
9.40	8	9.32	9.36	7	9.43	4	
9.45	2	9.43	9.43	1	9.44	0	
9.50	4	9.46	9.46	3	9.49	0	2
9.55	7	9.48	9.49	8	9.57	1	

Average = $\frac{65}{12}$ = 5.4

The average wait is reamrkably low, because of the fortunate circumstance that average time spent with the doctor is less than 5 min., and patients arriving early are able to take advantage of the doctor's availability.

Carry out this simulation for yourself.

Example (II) (An alternative method of approach to the problem in Example I)

Assume that in any period of 1 min., the probability that a patient arrives is $\frac{1}{5}$. If there is no queue, he goes in to see the doctor who takes 5 mins. to deal with him. Choose 60 random digits - let 9 or 0 indicate the arrival of a patient in the corresponding 1 min. period. The results may be shown in a chart.

Time period	1-2 3-5 6-7 8-9	8-15	16-20	21-25	26-30	31-35	36-40	41-45	46-50	51-55	56-60
R. digit	36981 94396	74184	16685	87103	93253	37526	69755	90542	39566	66324	13577
Patient arrives	P1 P2 P3			P4	P5		P6	P7 P8	P9 P10	P11	
W/M doctor (5 min unit)	3 (P1) 5 (P2) 11 (P3) 18			44 (P4) 49 (P5) 54			57 (P6) 61 (P7) 64 (P8) 52 (P9) 57 (P10) 62 (P11) 67				

Waiting times :- P1=0, P2=2, P3=6, P4=0, P5=3, P6=0, P7=1, P8=5, P9=5, P10=8, P11=10

Average = $\frac{38}{11}$ = 3.5 min

Repeat this simulation - compare your results with those given.

Examples (I) and (II) illustrate 2 different types of simulation. - In (II) we consider the situation at successive equal intervals of time and investigate which events occur in each time period, whereas in (I) we look ahead to the next event due, and the simulation moves in irregular time steps. Both methods are used in practice, although most of the examples in this booklet are of type I. Further suggested simulations are given below.

References

Selkirk, K. Random models in the classroom, Mathematics in School 2,6 Nov.1973 (and Jan., March 1974)

Mosteller, Rourke, Thomas. Probability and Statistics, Addison Wesley, 1961.

Some Suggestions for Simulation Exercises

Pupil worksheets based on some of these ideas are given in Section 4, pp 34→
Theoretical results for some are given in Section 7, pp 104→

1. Radioactive dalekium isotope has a half-life of 10 min. i.e. of a sample of 100g of Da, only half will remain 10 min. later, the rest having disintegrated, while after 20 min. only 25g remain.

(a) We can simulate the process by tossing 100 pennies, eliminating all which fall heads, tossing the remainder, and continuing, eliminating all which fall heads at each stage. Plot a graph of number remaining against time - this will simulate the radioactive decay process, each stage representing the half-life of the substance. Try this either with groups of coins, or tossing one coin repeatedly.

(b) Use a table of random digits instead, with p (decay) = 0.2
(this corresponds to a half-life of 3.1 units, since $0.8^{3.1} \approx 0.5$)
Use '1,2,3,4,5,6,7,8' = "success", '9,0' = "failure" - i.e. 'decay'

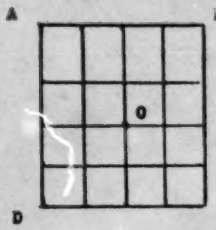
2. (Based on Selkirk, 1974)

100 samples from the same source are tested. Of these ¼ are contaminated by bacteria. Estimate the total number of bacteria in the contaminated samples (some of which contain more than one bacterium).

To simulate, choose pairs of random digits; these represent bacteria occuring in samples nos. 00-99. Some samples will be 'allocated' a bacterium more than once. Continue allocating until just half the samples are contaminated; how many allocations are needed?

(Theoretical value is $69.3 = 100 \log_e 2$)

3. Radiation shielding (atomic pile); 2 dimensional version.



A neutron starts off at O, in the diagram, which represents an atomic pile, O, surrounded by a region of shielding (the interior of ABCD'. It moves "N", "S", "E", "W", with equal probabilities (p = ¼) moving once every second.

(e.g. Take '1,2' = "N"; '3,4' = "W"; '5,6' = "S"; '7,8' = "E";

(ignore '9' and '0')

At the end of each move if it is still inside the shielding (i.e inside ABCD) it may be absorbed (with probability P_A). If it is not absorbed, it continues to move randomly. [For a simpler version suppose a neutron has a 'life' of 5 moves before it is absorbed].

(a) Trace the progress of 10 neutrons through the pile, each one starting at O. Record how many 'escape' by reaching the surface ABCD before they are absorbed. Take P_A = 0.1, say, ('1' = 'absorption', '2-9,0' = 'continue') [In a trial run, I found 4 absorbed, 6 escaped].

(Use random digits in pairs, one to give direction N,S,E,W, the other to decide whether particle is absorbed or continues. As an alternative, work with a partner, each using 1 digit from different parts of the random number table; one person determines the direction of the motion, the other whether or not the neutron is absorbed).

(b) Some suggested modifications:-

(i) Improved quality shielding - take P_A = 0.3

(ii) An extra layer of shielding of same quality (P_A = 0.1 but neutron does not escape until it has moved right outside ABCD). (see NEUTRON, p 98)

4. Server problem. (Adapted from Mosteller et al 1961)

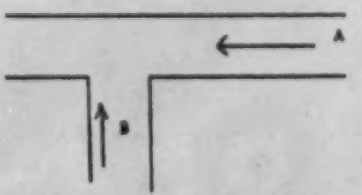
In a ten second period, a customer appears at a counter to be served with probability $p = \frac{1}{2}$. If others are ahead of him at the counter he lines up in the queue, otherwise the server serves him and takes 20 seconds to complete the service. In the 10th period of 10 seconds, what is the average number in the queue, if the process starts with no customers at the counter? Use a coin to carry out the experiment 5 times and get the average number. Also get the average number served at the close of the 10th period of time. Alternatively use a table of random digits with 0,1,2,3,4 = "customer arrives"; 5,6,7,8,9 = no customer. Example: C_X stands for a customer who arrived in the Xth time period:-

Time period	1	2	3	4	5	6	7	8	9	10
Arrivals	C_1	C_2				C_6	C_7	C_8	C_9	C_{10}
Being served	C_1	C_1	C_2	C_2	-	C_6	C_6	C_7	C_7	C_8
In queue	-	C_2	-	-	-	-	C_7	C_8	C_9	C_9, C_{10}

Total served = 4; number in queue in 10th unit = 2.

(see SARV, p90)

5. Road Junction The diagram shows a one way lane A joined by a feeder road B



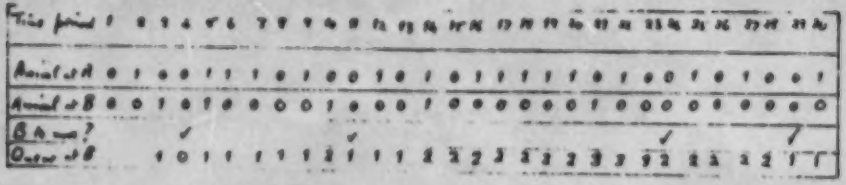
(a) In each 10 second period a car approaches from A with prob. P_A and independently, a car approaches from B with prob. P_B. Cars at B can exit only if there are 2 successive gaps in the A traffic; otherwise they form a queue.

Simulate, using $P_A = \frac{1}{2}$ (coin) and $P_B = \frac{1}{6}$ (dice). (Or $P_B = \frac{1}{5}$, random numbers 0,1 "success", 2-9 "failure").

Continue the simulation for 60 time periods (= 10 min)

What is the average length of Q at B?

Example



(b) Modifications:

Introduce traffic lights which change every 20 seconds (2 time periods of 10 seconds). Since traffic on A is 3 times as dense as on B, it may seem fair to make green for A=60 seconds, and green for B=20 seconds alternately (keep $P_A = \frac{1}{2}$, $P_B = \frac{1}{6}$ or $\frac{1}{3}$).

Experiment with other time ratios also.

(c) What happens if P_A and P_B are varied? (A has priority as the major traffic flow but suppose P_A and P_B are nearly equal. What then?)
(i.e. RBJ'N, p 100).

6. Lift

An automatic lift serves 3 floors A,B,C. In any one time period of 10 seconds, the probability of arrival of a passenger at A,B,C is the same ($p=\frac{1}{3}$, say) and each passenger is equally likely to go to either of his two possible destinations. (Suppose the lift takes 10 seconds to move one floor up or down)

i.e. $P(A \to B) = \frac{1}{6} = P(A \to C) = P(B \to A) = P(B \to C) = P(C \to B) = P(C \to A)$

Use coins (or use random digits, changing p to $\frac{1}{10}$).

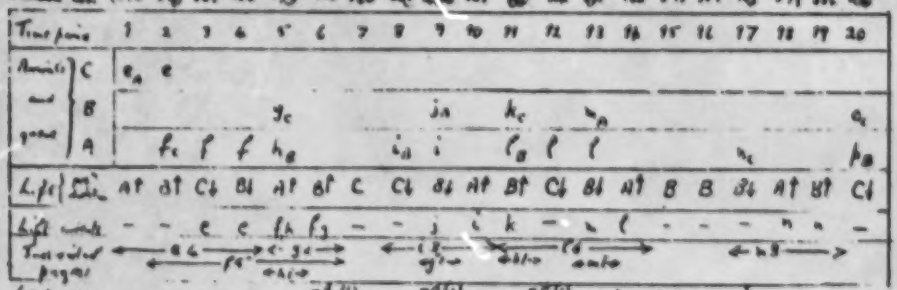
C
B
A

Note

Arrivals at A,B,C are independent, so { 3 choices of random digit / 2 tosses of coin triple } must be made

(Customers arrive at the beginning of each time interval, and enter lift at their floor if empty, or going in their direction).

I used random digits e.g. '1' = passenger at A for B, '2' = passenger at A for C etc. to obtain the table below. (Individuals are denoted by e,f,g, etc and the destination of each is shown as a suffix, e_c).



('1's & '2's entering lift; C → A(1) B → A(1) A → B(1) going downstairs).

(b) Possible modifications:-

(i) more realistic probabilities for destinations.

e.g. A→B, A→C } equal, (say p = $\frac{1}{10}$ for each).

C→B, B→C } smaller than the alternatives, (say p = $\frac{1}{30}$ each).

C→A, B→A } more likely (say p = $\frac{4}{30}$ each).

e.g. 00,01 = C→B

02,03 = B→C

04, 05, 06, 07, 08, 09, 10, 11 = C→A

12, 13, 14, 15, 16, 17, 18, 19 = B→A

(ii) Lift returns to ground floor A when not in use.

(iii) Probability of arrival of passenger at B is small (uses stairs), (say $\frac{1}{8}$ or $\frac{1}{10}$),

at C is, say, $\frac{1}{4}$ or $\frac{1}{8}$ (some may walk down);

at A is large, say $\frac{1}{2}$ or $\frac{3}{4}$ (going up).

7. Pinball Machines (Guile, C.A, 1979)*

(a) A simplified pinball machine is shown in the diagram (i). A ball enters at A where it meets a deflector; it is then equally likely to get to C (where it is lost) or to B where it meets a similar deflector. From B it moves either to C ($p = \frac{1}{2}$) or to D ($p = \frac{1}{2}$). From D (a "reflector") it moves back to A ($p = \frac{1}{6}$) or is 'absorbed' at C ($p = \frac{5}{6}$).

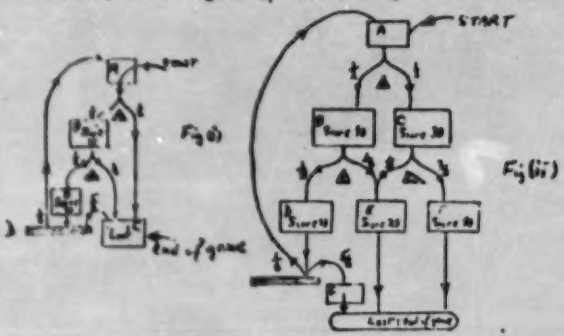
A simulation may be performed with a die (e.g. at D, throw a '6' to go back to A), or with random digits; either, for simplicity, change the probabilities at D to $\frac{1}{5}$ and $\frac{4}{5}$; e.g. 0, 1 - "back to A", 2-9 - "go to C" or use the digits 1,...6)

An example for the second case (random digits 1, ...6) is given below:-

A { 1 2 3 → left B ; 4 5 6 → right C

B { 1 2 3 → left D ; 4 5 6 → right C

D { 6 → left A ; 1 -5 → right C



Digits	4	3	6	2	2	2	4	3	5	6	1	5	6	1	2	4	5	4	4	1	2	5	3	2 .
Digits	4			3	6		2	2	2		4			3	5		6		1	5		6	1	2	4
Position of ball	A	C		A	B	C	A	B	D	C	A	C		A	B	C	A	C	A	B	C	A	C	A	B	D	C	.E

(see also p. 156)

(b) More complex machine diagrams may be considered, and 'rewards' associated with each of the 'states' A,B,C,D etc., as in "Amusement" Arcades. An example of such a diagram is given in (ii) above, and is considered later. (pp 90, 161).

(* Based on an idea of Miss C.A. Guile, undergraduate student at Keele).

8. Christmas Tree Lights

A set of Christmas tree lights contains 10 bulbs, each of which will fail during any given year's use with probability $\frac{1}{36}$. I have 2 spare bulbs, and when 3 bulbs have failed, the set will be useless, as further spares are unobtainable. How many years can I expect it to last?

Roll two dice. 'Double six' represents the event 'bulb fails'. 10 throws of two dice represent 1 year's use - continue until 3 bulbs fail. If random numbers are used, take probability of failure as 0.03 (e.g. use 00, 01, 02 to represent 'bulb fails').

Specimen results (using table of p 14 reading from top L)

Simulation number	Year	1	2	3	4	5	6	7	8				
(1)	Failures	1 (02)	0	1 (02)	0	0	0	0	2 (02,02)				(8 years)
(2)	Year	1	2	3	4	5	6	7	8	9	10	11	12
	Failures	1 (01)	1 (00)	0	0	0	0	0	0	0	0	0	1 (01) (12 years)
(3)	Year	1	2	3	4	5							
	Failures	1 (02)	0	0	1 (01)	1 (02)	(5 years)						

Average = $\frac{25}{3}$ = 8.3 years

(Theoretical result: 11 years - see p. 134)

Now repeat the simulation for a set containing 20 bulbs, other factors remaining the same. (Theoretical result 5.6 years, see p 135)

9. Petrol Pumps

A garage has a single petrol pump. Cars arrive for petrol at an average rate of one per minute, and take one minute to serve. If the pump is already occupied, they go elsewhere. Consider time units of 10 seconds; service time is 6 units, probability of arrival in one time unit is $p = \frac{1}{6}$

i.e. average gap between vehicles is 6 units (in this case, the same as the service time).

Simulate for 100 time units and estimate how many customers are served and the % who go elsewhere.

Now repeat, (i) reducing the average gap between vehicles to 3 units ($p = \frac{1}{3}$)

(ii) Supposing that <u>two</u> petrol pumps are installed (consider <u>both</u> traffic densities).

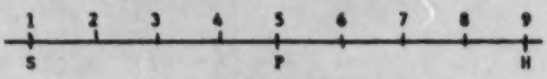
We can use single random digits 0 ⟶ arrival (or a die)

1,2,3,4,5 ⟶ no arrival (ignore 6,7,8,9)

Random number	49487			52802			28667		62058			78228			47041			18519		17889	
Time period	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Arrival	-	-	-	-	1	-	-	-	①	-	-	-		1	-	-	-	-	-	-
Service in progress?					✓	✓	✓	✓	✓	✓				✓	✓	✓	✓	✓	✓	

3 arrivals, 2 served

10. <u>Random walk</u>



A drunken man leaves a public house P at 'S' in the figure. His home H is at '9' and if he arrives there his wife will put him to bed. At '1' is a Police Station S; if he arrives there he will be put in a cell overnight to sober up. He takes half a minute to cover each unit of distance, at the end of which he falls to the ground and takes another half-minute to get up. Each time he begins to walk he is equally likely to go in either possible direction.

Use random numbers to simulate the process until he is 'absorbed', either at S or H, doing this several times. Investigate whether he appears equally likely to arrive at S and H, and on average how long he takes to get there.

Modify the simulation, supposing that the street runs downhill from S to H, so that he is more likely at each stage to walk towards H (e.g. take $p = \frac{4}{5}$ instead of $\frac{1}{2}$). See the programs DRUNK (p84), WALK (p85), and p121.

(Random walk models have been used extensively in studying the diffusion of gases. See also the 'neutron escape problem' pp22,98 , and Brownian motion, p33).

A reference for the theory of random walks is Dynkin, E.B and Uspenskii, V.A.(1963). Random Walks, D.C. Heath

11. Beetle

In the game of 'Beetle' the component parts of the creature are drawn when the appropriate number is thrown with a die. It is necessary first to throw a '6' in order to draw the body of the beetle, then, in any order, the following must be thrown - a '5' for the head, '4' for the tail, and four '3's' for the legs. (The anatomical details are conventional rather than exact! Although in some biologically more correct versions the beetle has six legs, this makes no material difference to the discussion which follows).

Once the '5' has been thrown, two '1's' for the eyes, and two '2's' for the 'feelers' must be thrown (again in any order). Players roll in turn, the winner being the one who first completes the drawing of the beetle.

(i) A complete game of 'Beetle' may be simulated using random numbers (e.g. with single digits, ignoring 0,7,8,9). We shall consider some theoretical results for this game later (pp 165ff); see also the computer programs BEETLE (§79), BEETLW (§774), BEETLN (§106).

(ii) Consider also a related simpler problem, that of throwing two '1's' and two '2's' in any order, with a die, and its extensions to three '1's' and three '2's', and, finally, four '1's' and four '2's'. ("In the game of 'Octopus' a '6' must be thrown for the body, followed, in any order, by four '1's' (for each left leg) and four '2's' for each right leg.....").

The theoretical results for the expected number of throws once the initial '6' has been obtained, are respectively 16.5 , 23.6 , and 30.6 - see p 116.

We illustrate a simulation of the simplest case, using single digits and ignoring 0, 7, 8, 9 :-

38458 49207 62358 41532 38847 53916 .....

2 2 1 1

19 throws are needed (cf. 16.5). See also the programs FIVE (p76), COLLGEN (§78).

12. Lucky 7

A die with 20 faces (an icosahedron) has a single digit on each face - each of the digits 0,1,2, ... 9 appears twice on the die. A game is played as follows:-

"Throw the die repeatedly until a '7' appears. If you need fewer than 3 throws you are lucky; if more than 15 you are unlucky".

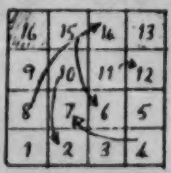
We use a table of random digits to simulate the process of throwing a '7'.

387 45 47 627 28856 28382 60639 5137 0 7 0091 58261 7 0135 88259
32519 91993 59364 83994 5987 3

Observed waiting times are 3, 4, 3, 19, 2, 10, 33, average $\frac{74}{7} = 10.6$
(see p 117).

13. Snakes and Ladders

A modified snakes and ladders board is shown in the diagram. A player starts on square 1. If he throws an even number he advances 2 squares; if an odd number is thrown, he advances 1 square. (A cubical die, or a table of random digits, may be used).



If he lands on square 16 he wins, if on 13, he loses.

Some specimen games follow:-

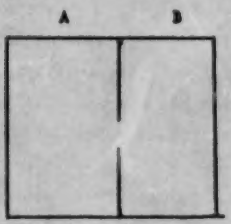
Initial position 1 2 3 5 7 14 16 1 2 3 5 7 9 12 14 6 7 14 16 1 3 5 7 14 16
Move 1 1 2 2 1 2 1 2 2 2 2 2 1 1 1 2 2 2 2 1 2

Position 1 2 7 14 6 14 6 7 14 6 7 9 12 12 1 2 7 9 12 13 1 3 7 9 2 ....
Move 1 2 1 1 2 1 1 1 1 1 2 2 1 1 2 2 2 1 2 1 2 1 .....

Result - won 3, lost 2. (c.f. p 137)

When 30 games were played on the board by a group of children, there were 22 wins and 8 losses (so p(win) estimated as 0.73).

14. A simulation model of gas diffusion (due to Ehrenfest), can be constructed as follows:-



Suppose a large number n of molecules is initially in compartment A, and that B is empty. A molecule is selected at random and transferred from the compartment it is in, to the other compartment.
(Initially the movement will be almost entirely in the direction A → B, but as soon as there are molecules in compartment B, there arises the possibility that one of them will be transferred back to A).
In the long run we should expect the molecules to be roughly evenly distributed between the two compartments. The simulation illustrates the progress towards this equilibrium state.
(For a given short interval in the distant future, there is a very small probability, 2^{-n}, that all the molecules will at that time spontaneously congregate in compartment B. But although each molecule is free to move independently of the others, statistical theory shows that the most probable distributions are those in which the proportions of molecules in A and B are nearly equal).

Doing the Simulation:-
A. The "Ehrenfest Bingo Game". $216(=6^3)$ squares on a board are each marked with a red, a blue and a green digit, so that each square corresponds to a possible throw of 3 coloured dice.
e.g. Red '2', Blue '1', Green '6' corresponds to -

The game begins with each square empty - as it progresses, squares may be covered by counters.
(i) Roll the 3 dice. Find the square corresponding to the result. If there is no counter on that square, put one there; if there is one there, remove it.

(ii) Keep a tally of the number of empty squares on the board (initially 216) - by adding or subtracting one at each move.
(iii) The number of empty squares on the board will initially decrease towards 108 (equilibrium) and then oscillate around that position. It is unlikely ever to fall below 96 (we might expect this about once in 370 throws). If it does, **BINGO**!

B. Random digits can be used instead of dice e.g. number a board 1 - 200 (or 1 - 1000) - and select 3 digit numbers.

C. A computer simulation may be used - output may either be the current number (n) of molecules in compartment A, or a graph showing the change in n with time. (c.f. programs EHR, EHR1, pp.86,87).

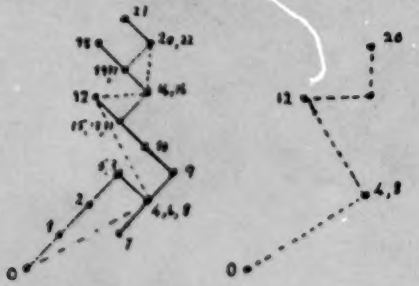
15. Brownian Motion

Small particles of colouring matter suspended in a liquid can be seen to move irregularly and unpredictably; this motion, first noticed by R Brown (1827) in pollen suspensions, provides indirect evidence for the atomic theory of matter. (The particles are responding to 'bombardment' by surrounding molecules). The motion is similar to a three-dimensional version of the random walk problem (p. 28), and may be simulated (in 2 dimensions) as follows:-

Use two independent series of coin-tosses (or odd/even random digits) to decide the 'sense' of motion parallel to the x and y axes.

Example

Coins	HHHH	THTH	HTTT	HHTH	TTHH	TH	x movement
	HHHT	HTTH	HHHH	THTH	HHTH	HT	y movement



The numbers on the dots indicate the time. Here every 4th position is joined (by the dotted line); a track similar to that of a Brownian particle is produced, (just as a photographic record of the motion of such a particle would indicate only a small fraction of the large number of individual movements it makes). As in the linear (1 dimensional) case, theoretical results predict the average distance from the starting point after n steps to be proportional to $\sqrt{n}$. (cf. progress p 85, WALK and p 91, BROWN).

SECTION 4

Pupil Material

Some suggested worksheets for younger pupils †

These suggest how some of the ideas of the preceeding chapters may be presented to pupils as simulation 'games' in a fairly concrete way, using either dice/coins or random digit tables to provide the element of chance where appropriate.

Teachers may wish to adapt them to their own circumstances or use them as they stand. (Photocopying, or other reproduction of these pupil worksheets by teachers for use with their own pupils is expressly permitted).

A table of random digits is given at the end of Section 2 and simple methods of generating random digits are also discussed in the note which follows that table (pp15-16).

Contents

		see pp.
1.	Crossing the road: Green X	35 (74)
2.	Petrol pumps	36 (27,81)
3.	Pinball	38 (26,90)
4.	Rumour	39 (1,89)
5.	Christmas pudding	41 (1)
6.	Radioactive waste	42 (21)
7.	Collector's problem	44 (1,77)
8.	Christmas tree lights	45 (27,111)
9.	Birthdays	46 (70,98)
10.	Road-up	48 (1,99)
11.	Road junction	50 (26,100)
12.	Inflation	52 (21)
13.	Atomic pile	53 (22,98)
14.	Ehrenfest 'Bingo' game (Gas diffusion)	54 (31,86)
15.	Drunk!	55 (28,84,121)
16.	Foxes and rabbits	57 (95)
17.	Insect-borne disease	59

† see p(ii)

Work Sheet1 GreenX (Crossing the Road)

Problem

A pedestrian wishes to cross a busy road from X to Y.
Traffic is coming from both directions L and R and so he can cross safely only if there is no vehicle in either of the road sections AY, XB in the diagram (XB = AY = 100ft., say).
Traffic arrives randomly from both directions; the probability that there will be a vehicle in AY during a given period of 10 secs (the time it takes for the pedestrian to cross) is p_L.
Similarly the probability that there will be an arrival from the right in XB, is p_R. What is the chance, in any time period of 10 sec, that he will be able to cross?

Simulation

Carry out the simulation as follows:-

(a) Suppose $p_L = p_R = 0.3$ (light traffic)

(i) Choose a random digit. If it is 1,2, or 3, place a counter in the section AY to represent a vehicle. If it is 0, or 4-9, leave AY empty.

(ii) Choose another random digit. Again if it is 1,2,or 3 place a counter in the section XB, otherwise leave XB empty.

(iii) If both AY and XB are empty record a successful crossing. Otherwise record 'unsuccessful'.
Carry this out 20 times (representing 20 x 10 sec time intervals), to find the number of successful and unsuccessful crossings.
Estimate the chance of a successful crossing.

(b) Repeat the whole simulation with $p_L = p_R = 0.7$ (rush hour) e.g. let '1,2,3,4,5,6,7' represent 'vehicle arrives', '8,9,0' = 'no vehicle'. You will need two separate random digits to decide what happens in the two road sections.

(c) Find out how to calculate the probabilities of a successful crossing in cases (a) and (b) above, and compare with your results.

Worksheet 2

Petrol Pumps

Description

Cars are arriving at a petrol station which has only one pump in operation. If there is a car already at the pump, they do not queue but go elsewhere and are lost as customers.

We suppose each car takes 60sec. to serve, and consider time units of 10 seconds, so that the service-time for a car is 6 'units'.

Suppose cars arrive at an average of one a minute (so that the probability p of an arrival in one time 'unit' is given by $p = \frac{1}{6}$).

(In this case the average gap between vehicles is 6 units, equal to the service time).

Problems

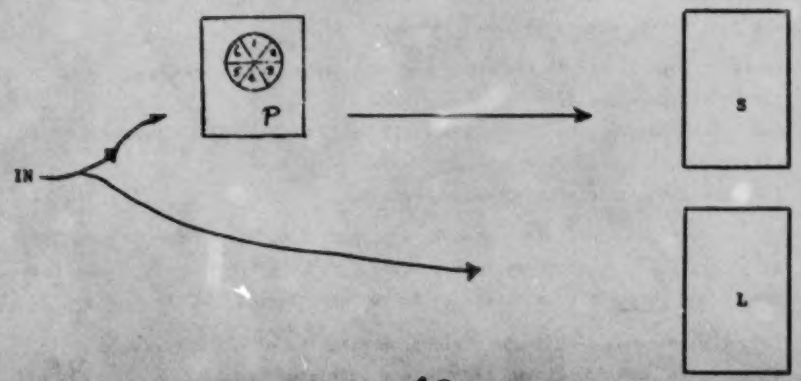
a) What proportion of customers will be served, and what proportion will go elsewhere?

b) How would the answer to (a) change if the traffic density is doubled? (i.e. 2 vehicles arrive per minute; $p = \frac{1}{3}$, where p is the probability of an arrival in a single time unit).

c) Suppose the garage installs <u>two</u> petrol pumps. What difference does this make to the situations described in (a) and (b)?

Method

Use two sheets of paper labelled S and L (to keep a record of the customers served and customers lost respectively), and another sheet P , marked with a 'clock' (a circle with 6 subdivisions, as shown) to represent the petrol pump. A counter placed on P will represent a car being served. In each time unit (10 sec) the clock finger moves on one subdivision. (Use a pencil or another counter as the finger). When the clock shows '6', the car is ready to leave.



(i) At the beginning of each time interval, throw a die.

(ii) If '6' is obtained (representing 'car arrives') a counter is taken.

(iii) A counter representing a new vehicle may be placed on sheet P if P is empty ("pump free"), otherwise it is placed on L ("customer lost").

(iv) If there is a counter on P, the clock 'finger' is moved round one place. When the finger shows '6', the car is ready to move to S ("customer served").

(v) End of time unit.

Continue from step (i) until 100 throws of the die have been made. At the end, record the number of counters on S and on L (ignore counters still on P).

For (b):- alter step (ii) to give "vehicle arrives if dice shows '5' or '6'."

For (c):- use two cards P_A and P_B to represent two pumps

In (iv) the clock 'fingers' for any counters on P_A or P_B are moved one place clockwise in each time unit.

Modifications

(d) Use more cards to represent installation of extra pumps.

(e) 'Speed up the service time' by installing a high-speed pump (i.e reduce the petrol pump 'clock' to 4 subdivisions)

(f) Alter the density of traffic - e.g. by using

'4', '5' or '6' = vehicle arrives ($p = \frac{1}{2}$ - i.e. 3 vehicles/minute)

or '3', '4', '5' or '6' = vehicle arrives (4 vehicles/minute).

Note

Tables of random digits may be used instead of dice.

E.g. '9' or '0' = "vehicle arrives" gives a traffic density of 1 vehicle per 50 seconds, ($p = \frac{1}{5}$).

Worksheet 3 Pinball

Fig(i) is a pinball machine - you sometimes see them in Amusement Arcades.

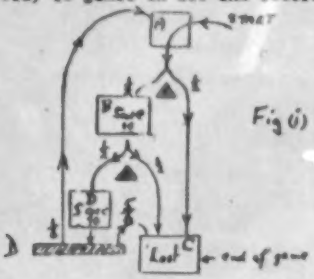
How to Play

You need a counter (or some other 'marker') to show where the (imaginary) "ball" is, and a die to decide where the ball goes next. Keep a record of your score in each game.

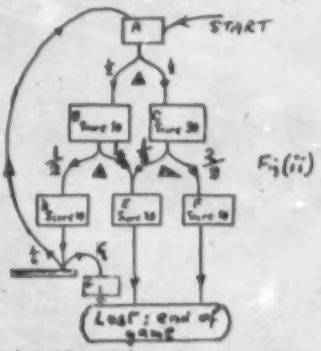
1. Place the counter on square A (start).
2. Roll a die. If the result is even, go to square C (end of game).
   If the result is odd go to square B and add 10 to your score.
3. From B - roll a die again. If the result is even, go to square C (end of game).
   If the result is odd, go to square D and add 10 to your score.
4. From D - roll a die. If you get a '6', go to square A and start again.
   If you get anything else, go to square C (end of game).

Record your score for the game. What do you think your average score in a lot of games would be?

Play 10 games in all and record your score for each. Work out the average.



Fig(i)



Fig(ii)

What to do Next

(A) Use the diagram of the more complicated pinball machine shown here. Write down the rules you will use to decide where to go next from each box, and how to alter your score at each stage. (To help you - you might decide at C:- "throw a die. Score of '1' or '2' → go left to E; score of '3', '4', '5' or '6', go right to F").

Play one or two games and record your score for each.

Estimate your average score in a lot of games.

Now try playing (say) 10 games and work out your average score for all 10.

(B) Make up your own pinball machine diagram and play some games on it. Don't make it too complicated; remember you have to predict what the average score in a game will be!

Worksheet 4

Rumour

A rumour spreads among a group of 6 friends like this:-
When one person hears the rumour he telephones one of the other 5, chosen at random, to pass it on. When he has done this, he takes no further interest in the rumour. The second person passes on the rumour in the same way to just one other person chosen randomly; if the message is passed to someone who has already heard it, both individuals stop talking about it, and no further people are informed.

Problems

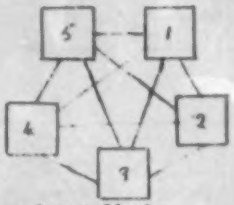
Is it likely that all 6 will get to hear of the rumour?
On average, how many of the 6 will get to hear of it?

Simulation

We can choose one person at random in a group of six by rolling a die.

Method

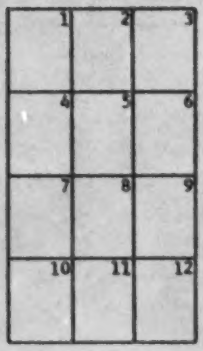
Use a copy of the diagram and a supply of counters.
Roll a die; whatever number it shows, place a counter on that square (to show a new person has been told the rumour), unless the number is the same as the previous one (he wouldn't tell the rumour to himself!).
If you get two counters on a square it means someone has heard the rumour twice - so it dies out. (A person is allowed to tell the rumour straight back to the person who told him - perhaps he is very forgetful!).
At the end, record how many squares have counters on (i.e. how many people have heard the rumour).
Repeat the experiment several times - put the results in a table like this



Expt. number	1	2	3	4	5	6 .....
No. of people hearing rumour	4	6	2	5	.....	

Variations

Suppose now there are 12 people in the group. Use 2 dice. If the first one shows 'even', use the number on the second die to decide who hears the rumour next. If the first one shows 'odd', add 6 to the score on the second die in order to decide. (Ignore cases where a person tells the rumour to himself - i.e. you get the same result twice running).



Put a counter on a square to show that the person has heard the rumour.
Stop when you get two counters on the same square.
Repeat the experiment several times and make a table as before.

Experiment number	1	2	3	4	5	6	7 .....
No. of people hearing rumour							

Worksheet 5

Christmas Pudding

Description and Problem

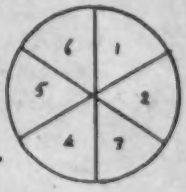
Suppose a Christmas pudding is shared between six people so that all have equal portions. Hidden in the pudding there are six coins. Is it likely that every person will get a coin? How many people, on average, will get no coins? How many will get more than one?

Simulation

Use a dice to decide, for each of the 6 coins, which portion of the pudding it goes into.

Method

(a) Use a copy of the circle shown divided into six sectors representing the six portions. Roll a die, and place a counter on the sector corresponding to the number shown. Continue until 6 counters (6 coins) have been placed in sectors.



How many sectors are empty ("have no coins")?

How many sectors have more than one counter ("more than one coin")?

Repeat the experiment several times and record your results in a table like this:- (Each column represents a new experiment)

Expt. no.	1	2	3	4	5 ...
No. of empty sectors					
No. of sectors with more than one counter					

Variations

(b) Use random digits to simulate sharing a 10-coin Christmas pudding between 10 people.

Choose 10 digits in turn and place a counter on the corresponding square each time.

1	2	3	4	5
6	7	8	9	10

(c) Use 2 dice - or instead use one die and a penny - to simulate division of a 12-coin Christmas pudding between 12 people.

Method

Roll the first die. Then roll the other die (or toss the penny). If the second die shows even, (or the penny shows 'H') add 6 to the score of the first die; otherwise add 0 to the score of the first die.

Notice that the proportion of people who get no coin at all is about the same in all three cases (a), (b) and (c).

Worksheet 6

Radioactive Waste

Description

Suppose a new radioactive substance ('Dalekium') has been discovered with a "half-life" of 10 minutes. (This means that, if at first there is 100g of the substance, after 10 minutes only 50g will remain, and after a further 10 min., only 25g will be left - and so on - because it 'decays' to form other substances).

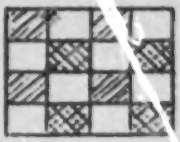
Problems

If we start with 100g, how much will be left after one hour?

How long will it take for the amount present to fall below the "danger level" (of 0.001g remaining)?

Simulation

We simulate the process of decay by scattering counters at random on a board, so they fall on a shaded square with probability ½ (each counter which falls on a shaded square corresponds to an atom which disintegrates during a 10 min period). Those counters which have fallen on a shaded square ("atoms which have disintegrated") are removed, and another '10 min cycle' of the simulation is performed, by scattering the counters again.



Method

Place a piece of A4 paper, with alternate squares shaded (like a chessboard - see fig.) inside a box lid. (A box in which A4 paper is supplied is ideal).

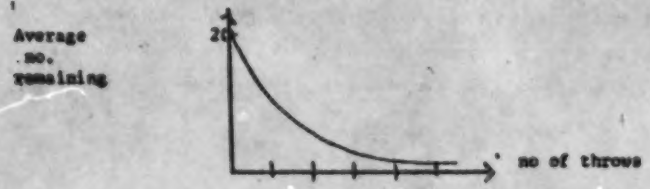
Place 20 counters (or 1p coins) in the box lid and shake them up (or side ways).

Eliminate any counters/coins whose centres lie on a shaded square.

Repeat until all counters are eliminated, and record your results at each stage:-

Time interval (measured in 10 min ('half-life') periods	0	1	2	3	4	5	6
No. of counters ("atoms") remaining	20	11	5	.....			

Repeat the experiment two or three times more, and work out the average number remaining after 1 throw, 2 throws, 3 throws.....
Plot a graph of average number remaining against number of throws.



It should look something like the smooth curve in the diagram - but since the number of 'atoms' in your simulation is small, and the amount remaining must be a whole number of atoms, the graph of results is likely to have 'kinks' in it.

Note,

100 g of any substance contains a very large number of atoms - about 100000000000000000000000 (23 zeros) in fact, so that the 'kinks' in the graph representing decay of an individual atom are too small to be noticed.

(Alternatively coins may be tossed and those which fall 'heads' removed at each stage. The method using counters is likely to be quicker and more convenient - and not so many coins are needed!)

Modification

Simulate the behaviour of another isotope which decays more slowly, (p (decay in 10 min period) = ½) i.e. ½ of the original amount is left after 10 minutes.

Method

Use the counters as above, eliminating only those which fall with their centres on a doubly shaded square (p = ½)

Again record results in a table, as above.

(If coins are used, before eliminating those which fell H, toss them again and eliminate only those which fall H a second time - $P_{HH} = \frac{1}{4}$).

Worksheet 7

Collector's Problem

A cereal manufacturer provides a small plastic badge with each packet of cereal. If there are 6 different types of badge and they are randomly distributed one to each packet, how many packets of cereal must I buy, on average, to obtain a complete set of the badges?

Simulation

(a) Roll a cubical die and take the number displayed as the number of the next badge collected. Continue rolling the die until all of the numbers '1,2,.....6' have appeared. Record the total number of throws required. (It may help to write a list of numbers 1 - 6 and cross them off as they appear.)

(b) Repeat several times for comparison. Find the average number of throws required.

Alternatives

(a) Simulate collecting a set of 8 objects by tossing 3 coins, until each of the possible 8 results has appeared. HHH HHT HTH HTT THH THT TTH TTT

(b) Simulate collecting a set of 8 digits by using a cube with corners labelled 1,2,.....8.
(Selection rule: "Choose the uppermost corner which is nearest to your eye. In the event of a tie, choose the right hand one").

(c) Simulate collecting a set of 12 objects using a cube with edges labelled 1,2,.....12.
(Selection rule: "Choose the uppermost edge which is nearest you; in the event of a 'tie', choose the right hand one").

(d) Simulate collecting a set of 10 objects (or n objects where $n \leq 10$) using random digits. (For $n < 10$ ignore all digits outside the appropriate range).

Worksheet 8

Christmas Tree Lights

Description and Problem

I have a set of Christmas tree lights containing 10 bulbs. The chance that any single bulb will fail during one year's use is $\frac{1}{36}$. I have two spare bulbs, so that when three bulbs have failed, the set will be useless - as spare bulbs are now unobtainable.

How many years can I expect the set to last?

Method

Place 10 counters on the table to represent the 10 bulbs, and have two others ready to represent the spares. For each counter in turn, throw 2 dice to decide if that bulb fails during the year. (If you get 'double-6' assume 'bulb fails' and remove that counter - it represents a 'dud' bulb). Continue until all counters have been tested; that represents one year's use. (The spares are not tested, as they are not in use).

Now, if you have spare bulbs left, make up the set to 10. If you cannot, the set is now useless - record how long it lasted. If you can make a complete set of 10, continue the experiment for another "year".

Try the whole simulation several times over, to compare results.

Modifications

(a) Use cheaper, poor quality bulbs - e.g. 'double-5' or 'double-6' = 'failure', (so that P(failure) = $\frac{1}{18}$).

(b) If the bulbs are cheaper we may be able to allow more spares - so allow 4 spares (i.e. stop when 5 bulbs have failed).

(**Example**

Year number	1	2	3	4	5
No. of bulbs failing during year	1	0	1	2	1

(set fails after 5 years))

(c) Use a 'super quality' bulb with a low probability of failure (say P = $\frac{1}{72}$) but only one spare bulb. (So that the set is useless after 2 failures).

(For P = $\frac{1}{72}$ use 2 dice and a coin. Bulb fails only if coins shows 'head' and dice show 'double-6'.

Alternately random digit tables may be used, e.g.

random digits 01 or 02 } ⟶ 'bulb fails': this gives P(fail) = $\frac{1}{50}$)

Worksheet 9

Birthdays

What is the chance that in a class of 30 children, there are two people with the same birthday? The answer is rather surprising.

(1) You might like to try it for your class - list everyone's birthday. Are there two the same?

Now use either (2) or (3) below:-

2. Using Dice

To make the problem simpler we will suppose that there are 12 equal months, each of 30 days. (This would give 360 days in a year - not far out).

(A) Roll 2 dice, to decide the birth month. Roll first one and then the other. Use the first die to decide whether the result is Jan-June ('1','2','3' on 1st die) or July - Dec ('4','5','6' on first die) and the second to decide which month, (from 1-6).

Example

If you throw '4' first time (July-Dec) and '2' second time, (Feb. or August), the result is August.

(B) Now find the date within the month. Roll 2 dice in succession.

(i) If the first die score is '6', roll it again.

(ii) Take first die score and subtract one.

(iii) Multiply the result by 6 and add score on second die.

Example

Scores are '3','5'. Date is (2 x 6) + 5 = 12 + 5 = 17(th).

Note. A score of '1','1' would give (0 x 6) + 1 - i.e. 1st of month and of '5','5' would give (4 x 6) + 6 i.e. 30th of month.

(Remember: score of '6' on first die is ignored - throw again).

(C) Write down the 'birthday' obtained in this way - in our case Aug 17th.

(D) Repeat this until you have 30 randomly chosen 'birthdays' (but stop if you get two the same before you have done 30).

Do the whole experiment several times.

What do you think is the correct value for N in the following statement?

"If there are more than N people in a room, the chance that two of them have the same day as their birthday is $> \frac{1}{2}$"

3. Using Random Digits

Suppose each of the days of the year is labelled, from 1 to 365 (we ignore leap years).

(A) Choose 3 random digits to give a 3 figure random integer R.
If $1 \leq R \leq 365$, R gives the date.
If $501 \leq R \leq 865$, take the date as $(R-500)$.
(If R lies outside these ranges, try again)

(B) Repeat this process to assign birth dates to 30 people (or until 2 birth-dates coincide).
Do the whole experiment several times.
What do you think is the correct value of N in the following statement?
"If there are more than N people in a room, the chance that two of them have the same day as their birthday, is $> \frac{1}{2}$".

Worksheet 10

Road Up

Description

Temporary traffic lights are installed to control one-way traffic when a section of road running North/South is under repair (see fig.).
Because vehicles cannot clear the one-way section instantaneously, there will be a lag between the halting of traffic in one direction, and the release, by a 'green' signal, of traffic in the other. The rate of flow of traffic in the two directions may differ - (e.g. at rush hour).

Problem

How should the timing sequence for operation of the traffic lights be adjusted for best results?

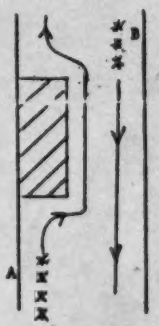
Simulation

Suppose the time lag between 'green' signals (to allow traffic to clear the one-way section) is 5 seconds, and that just two vehicles can pass through the section in 5 seconds. Suppose also that N-bound traffic arrives at the average rate of 6 vehicles per minute (½ every 5 sec.) and traffic travelling S, at 12 per minute (1 every 5 sec.).

(a) Consider first the traffic light pattern:

I	N bound lights green,	5 sec.
II	Both lights red,	5 sec.
III	S bound lights green,	5 sec.
IV	Both lights red,	5 sec.

and use 5 sec. as a unit time interval.



Method

In each time period:

(i) Toss one coin (for N bound traffic). If coin shows 'H' place a counter at A (to show "vehicle arrives").

(ii) Toss 2 coins (for S bound traffic).
If result is HH, 2 vehicles arrive at B.
If result is HT or TH, 1 vehicle arrives } at B.
If result is TT, no vehicle arrives }

(In brief, number of vehicles arriving = number of 'heads' obtained).

Place counters at B to represent vehicles.

(iii) If traffic lights are 'green', waiting vehicles may move forward. At most two vehicles may pass through the narrow section in one time period. Any which cannot pass through remain in the queue. (Those which have successfully passed through should be kept - or at least counted).

(iv) Traffic lights at A and B change according to the pattern given.

Continue the simulation for 24 time periods, by returning to step (i). At the end of this activity, examine the size of the queues which have formed, and the numbers of vehicles which have successfully passed through the narrow section in N and S directions.

(b) How change the traffic light pattern to:-

I	N green,	10 sec.
II	Both red,	5 sec.
III	S green,	10 sec.
IV	Both red,	5 sec.

and repeat the simulation. Is this pattern an improvement?

(c) Repeat for the pattern

I	N green,	25 sec.
II	Both red,	5 sec.
III	S green,	25 sec.
IV	Both red,	5 sec.

(d) Repeat for the pattern

I	N green,	15 sec.
II	Both red,	5 sec.
III	S green,	30 sec.
IV	Both red,	5 sec.

Which seems the most appropriate pattern?

Can you suggest a better one still? Try it out!

Worksheet 11

Road-Junction

Description

The diagram below represents a one-way traffic lane joined by a feeder road B, (such as might be found on a dual carriageway). In each 10 second period a car approaches from the direction A with probability P_A and, independently, a car approaches from B with probability P_B. Cars at B can exit only if there are two successive gaps in the A traffic; otherwise they form a queue.

Problem

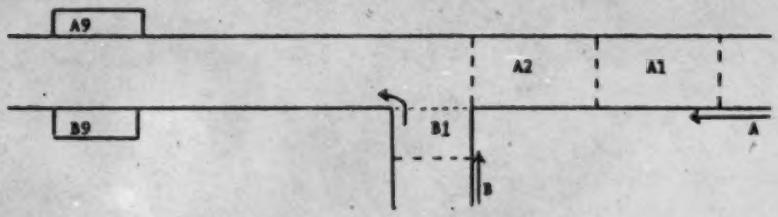
How does the queue length at B depend on traffic density?

Would installing traffic lights help?

Simulation

Take $P_A = \frac{1}{2}$, and $P_B = \frac{1}{6}$. We use a coin to decide arrival of traffic from A, and a die for traffic from B.

Method



Use a copy of the diagram, which represents a road junction with A and B traffic coming from the directions as shown. B traffic turns left into the traffic stream (if no A traffic is present - A has right of way). The regions A9, B9 are used to hold counters representing A and B vehicles, respectively, which have passed through the junction.

(i) If there is a counter in A2, move it to square A9 (A traffic which has gone past the junction).

(ii) If there is a counter in A1 move it to square A2 (traffic moving along main road).

(iii) Toss a coin. If the result is 'H', place a counter on square A1 to represent "car coming".

(iv) Roll a die. If the result is '6', place a counter on square B1 (vehicle arrives from B) - to represent a queue forming.

(v) If A2 and A1 are both empty, allow one car from B1 to leave the queue. (Remove one counter and place on square B9 (B traffic which has gone past the junction).
This corresponds to a 10 sec. period.

(vi) Go back to step (i), until you have done 30 steps of the simulation (i.e. '5 minutes').

At the end of the experiment, record results:

Number of vehicles from A (= no. of counters in A9)

Number of vehicles in B queue (= no. of counters in B1)

Number of vehicles from B (= no. of counters in B1 + number in B9).

Alternatively, write down the number of vehicles in B traffic queue (i.e. counters on B1) at each stage, and find the average queue length. (This gives a measure of the average waiting time for vehicles trying to escape into the main stream of traffic).

Modification

Suppose traffic lights are installed, so that 'A' traffic can move for 6 periods of 10 sec, and is then halted for 2 periods of 10 sec. to allow 'B' vehicles to move. *

Suppose 2 vehicles can pass through the lights in a period of 10 sec. when they are showing 'green'.

Method

(i) Toss a penny to decide arrivals at A. (Ignore A1 and put vehicle counters on A2).

(ii) Roll a die to decide arrivals at B.

(iii) Add vehicles to queues at A and B.

(iv) Reduce queues at A and B according to traffic light setting.
(N.B. queue can reduce by 2 if there are sufficient vehicles in the queue).

(v) Go back to stage (i) until 32 steps of the simulation have been carried out.

Compare number of arrivals at A,B (counters on A9 + queue A2 and counters on B9 + queue B1, respectively) - they should be similar to earlier findings.
Compare queue lengths and average queue lengths under the new system - is it an improvement?

---

*You may wish to experiment with other traffic light settings to see if you can improve further on this.

Worksheet 12

Inflation

For this simulation you need a calculator.

Problem (1)

The value of a certain house increases by 15% every year. How long will it take to treble in value?

Method

If the rate of increase is 15% each year,

after 1 year, £1 becomes £1 x 1.15 = £1.15

after 2 years, £1 becomes £1 x $(1.15)^2$ = £1.3225, and so on.

Use a calculator to find how many times you must multiply by 1.5 to convert £1 into £3.

Problem (2)

What rate of inflation will double the value of a house in 3 years?

(The answer is not 33½%).

Method

By trial $1.15^3 < 2$, so 15% is not enough.

Also $1.3^3 = 2.197$, so 30% is too much.

Use trial and error to find the % inflation needed.

Problem (3)

An isolated island has plenty of grass and a population of rabbits which increases at a constant rate. (The number of rabbit-births and deaths is proportional to the number of rabbits present - assuming that famine, disease and destruction by other animals do not intervene).

If the population increases 10 fold in 10 years, how much will it increase in one year and in 15 years?

Method

The same as for problem (2).

Problem (4)

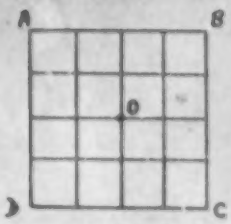
Radioactive dalekium isotope has a half-life of 10 min. i.e. of a sample of a 100g of Da, only half will remain 10 min. later, the rest having disintegrated, while after 20 min., only 25g will remain.

Find the amount remaining after 1 hour and also after 1 minute.

How long will it take for the amount to fall below the 'danger level' (0.001g remaining)?

Method

See (2) and (3) above.



Worksheet 13

Atomic-Pile (simplified version in 2 dimensions)

Problem

The diagram represents an atomic pile O, surrounded by a layer of shielding A B C D. Neutrons are emitted from the pile; the shielding is intended to absorb them before they leave the square A B C D.

How effective is the shielding?

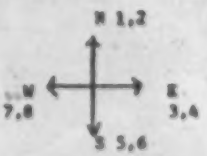
Is it thick enough?

Simulation

We assume that a neutron starts from O and each second moves exactly 1 unit either N, S, E or W, (or stays where it is), and that if after 5 seconds it has not escaped, by reaching the edge of the square A B C D, its energy is so low that it will be absorbed where it is.

(a) Start with a neutron at O. Choose a random digit and decide the movement of the neutron by these rules:

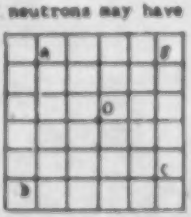
If digit is { '1' or '2', move N; '3' or '4', move E; '5' or '6', move S; '7' or '8', move W; '9' or '0', stay put }



Trace the progress of the neutron for 5 moves (5 digits) to decide whether or not it escapes. Repeat for 10 neutrons altogether.

What fraction of the neutrons escapes?

(If you can, pool your results with those of other pupils; you will get a better overall estimate since your group of neutrons may have been 'lucky' or 'unlucky').



(b) Repeat the simulation using a thicker layer of shielding (a 6 x 6 square) tracing 10 neutrons as before, (see figure).

(c) Repeat (b) but now suppose the neutrons to be more energetic with a 'life' of 10 sec., (so trace their progress using 10 random digits instead of 5).

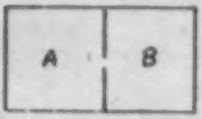
Is the thicker shielding effective?

Worksheet 14

Ehrenfest Bingo Game - a model of gas diffusion

Problem

Two equal compartments are separated by a partition in which is a small hole. (See diagram (i)). At first, A contains a large number of molecules of a gas and B is completely empty. Molecules in A tend to move into B; any which have moved into B tend to move into A again. After a long time period, we expect the molecules to be fairly evenly divided between A and B ("equilibrium").



(i)

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Board B.

(ii)

Simulation

We choose a molecule at random and transfer it to the other compartment. At first the movement will be almost entirely in the direction A ⟶ B, but gradually the rate of transfer in the other direction will increase as the number of molecules in B increases. We suppose there are 36 molecules in A to begin with.

Method

Each of the 36 squares on the board B is labelled to show the result of throwing 2 dice; e.g. (3,6) represents '3' on first die and '6' on second. (i,j (ii)).

At first B is empty (corresponding to the empty compartment B).

Each time the two dice are thrown a counter is placed on the corresponding square of B, unless there is already one there - in which case that counter is removed from B. The number of empty squares on B is recorded. At first this will decrease sharply, but as the 'equilibrium position' is approached, the rate of decrease will slow, until in the later stages random oscillation about equilibrium will occur. It is instructive to draw a graph of number of squares against number of throws (i.e. 'diffusion' against 'time').

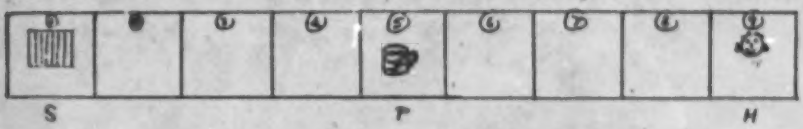
(cont'd. on p. 56)

Worksheet 13

Drunk!

Description

A drunken man leaves a public house P at '5' in the diagram below. His home H is at '9', and if he arrives there, his wife will put him to bed.
At '1' is a Police Station, S; if he arrives there he will be put in a cell overnight to sober up. He takes half a minute to cover each unit of distance, at the end of which he falls to the ground and takes another half a minute to get up. Each time he begins to walk, he is equally likely to go in either possible direction.



Problem

Is he more likely to end up in jail or at home?
How long, on average, does it take for him to get there?

Method

(a) The time between successive tosses of a coin represents one minute. If the coin falls 'H' he moves one step towards home (Right). If it falls 'T' he moves one step Left (towards jail): ($P_L = \frac{1}{2}$, where P_L = probability of moving Left: if you use a die, take 'odd number' = 'H' = move R, 'even number' = 'T' = move L).

(i) Start on square '5'. Place a counter there to represent the man.

(ii) Toss the coin and move the counter according to the rules.

(iii) If the counter has not reached square '1' or square '9', return to step (ii). If it has, record the number of moves taken and where he ends up (in a table like that shown below) and return to step (i) until you have performed the simulation several times.

Specimen Table of Results

Simulation Number	Finishing Point (S or H)	No. of Moves
1	S	10
2	S	4
3	H	10
4	S	18

Modifications

(b) Suppose the street slopes down hill from S to H, so he is more likely to move R at each step.

(With a die, take '5' or '6' = move L, so $P_L = \frac{1}{3}$

With a coin, toss twice and use 'HH' = move L; 'TT', 'HT' or 'TH' = move R, so that $P_L = \frac{1}{4}$).

As before, find whether he is more likely to end up at S or at H. When he starts from P, what is his chance of reaching H?

Note. Tables of random digits could be used instead:

e.g. in (a) Odd digit ⟶ "move L"; even digit ⟶ "move R"

in (b) Digit = '0','1','2' ⟶ "move L"; any other digit ⟶ "move R"

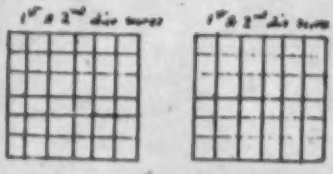
(so $P_L = \frac{3}{10} = 0.3$).

Worksheet 14 (contd from p. 54)

Extensions

(a) Use 3 dice and 216 ($=6^3$) squares arranged in 6 sets of 36.

(216 molecules in A at the start).



etc. .. ....

(3rd die tells you which square to choose; 1st & 2nd set you position in that square).

or

(b) Use pairs of random digits and a 10 x 10 square for B, or triples of random digits and a board numbered 1 - 200, (or even 1 - 1000).
(The latter is best as a class activity).

Worksheet 16

Foxes and rabbits

An island is thickly covered with grass and supports a large population of rabbits; let the number of rabbits at a given moment be X. If there were no checks on population growth, X would increase exponentially, (ie according to the "law of compound interest"), since the number of rabbits born in a period of time would be proportional to the existing population. Suppose, though, that a population of Y foxes also lives on the island, feeding on the rabbits. If we assume no rabbits die of old age, the rate of decrease of rabbits will be proportional to the product XY. Similarly, we assume that the rateof growth of the fox population is proportional to XY (more foxes, more cubs; more rabbits, more food for them). The foxes may die of old age, and then the rate at which they are reduced will be proportional to Y. We shall measure time in months, and choose the units of X and Y so that the equations determining X and Y become

$$\Delta X = \tfrac{1}{2}X - \lambda XY \quad \ldots\ldots\ldots (1)$$

(change in X per month = {increase due to births} + {Decrease due to foxes}

$$\Delta Y = -\tfrac{1}{2}Y + \lambda XY \quad \ldots\ldots\ldots (2)$$

(change in Y per month = {decrease due to deaths} + {increase due to births}

where $\lambda = \frac{1}{70}$.

It can be deduced from these equations that both populations will oscillate in number about a mean of 35, as shown in the diagram:-

(When there are a lot of rabbits, the foxes will multiply and so begin to reduce the rabbit numbers; as this happens, there will be less food for the foxes, so their numbers will diminish in turn, allowing the rabbits to increase again).

Simulation (adapted from Thompson, 1979*)

We use a 'stochastic' method (ie one based on chance) which mirrors the equations 1 and 2 in terms of probability. Because of the chance element, there is a possibility that one of the popu lations will die out.

Method

Use a piece of A4 paper with 32 rectangles, shaded alternately inside a box 11d. (cf Worksheet 6, p 43)

(* Thompson, J MT,An evolution game for a prey-predator ecology, Bull I.M.A. 15, 7 July 1979)

Take F black counters ('foxes') and R white ones ('rabbits') - suitable starting values are F=10, R=10.

1) Shake all the counters in the box lid.
   A counter is 'on' a square if its centre is inside the square.
   If any square contains more than 3 counters of any one colour, 'excess' counters are removed. ("deaths by overcrowding").
2) Any black counter which is 'on' a black square is removed (Fox dies, p = $\frac{1}{3}$)
3) Any white counter on a black square is replaced by two white counters (rabbit reproduces, p = $\frac{1}{3}$)
4) For each pair of black and white counters which are together on a white square, two black counters are added and two white counters are removed (two foxes are born, and two rabbits eaten)

   (Note. Step 4 is not repeated for black counters representing newly-born foxes)

The numbers of foxes and rabbits at the end of step (4) are recorded and the whole cycle repeated.

Note. In step 4 it is found experimentally that when X=Y=35 (the 'equilibrium" values) a board of 32 squares gives about 17 interactions between foxes and rabbits, agreeing with the value $\lambda = \frac{1}{70}$ in equations (1) and (2).

Worksheet 17

Diffusion of an insect pest (*)

The diagram (fig (i)) shows an area of land near a port P where an insect pest has been accidentally introduced. The area shaded diagonally represents barren ground, where the pest cannot live; the area marked with dots is the sea. The pest spreads by flying and is most likely to spread NE because of the prevailing wind.

Simulation

Infection spreads from an infected square to adjacent squares according to the scheme shown in fig (ii). Two dice are thrown; the sum of the scores indicates the next square to be infected (unless it is sea or barren land).

Method

1) Mark the port with the number 0 (generation 'zero') Throw the dice and determine the next infected square; mark this 1 (generation '1')
2) For each square infected by generation 'n' (ie 0, 1, 2, 3, 4.... n) determine the 'next square infected', using fig (ii) - mark all such squares generation '(n+1)'.
3) Repeat step (2) until n=20, say.

Note: many choices of 'next infected square' will be wasted because the square is already infected, or is barren or sea.

Modification

Tables of random numbers may be used instead, with fig (iii), to determine 'next square infected'. eg, if the 2-digit random number chosen is 43, then infection takes place [illegible]

Notice the effect of the 'corridor' between the mountain regions A and B, and also how slowly the infection spreads westwards.

(*based upon an idea of K Selkirk).

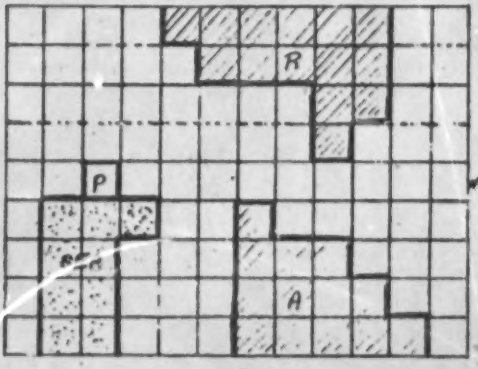


fig (i)

fig (ii)

6	5 or 3	7, 2 or 12
4		9 or 11
–	10	8

Infected square

Sum of scores on 2 dice

fig (iii)

91-00	26-90	1-25
81-90		41-55
76-80	66-75	56-65

Infected square

Section 5
Computer Simulation

In this section, which introduces the use of the computer in simulation, some acquaintance with the programming language BASIC will be assumed, though much of the material here should be intelligible even if you have no experience of writing programs in BASIC.

Computers usually have a random number generator available, enabling one to write such statements as LET X=RND which assigns to the variable X a (real) number randomly chosen from the interval $0 \leq X < 1$

(Strictly speaking, since numbers are stored in the computer to only limited accuracy, the value of X cannot vary continuously between 0 and 1, and X can only take one of a finite (though large) set of values in this range. cf. p 6. Further, the numbers are not truly random, being generated by a determinate process (see below). They are more correctly called 'pseudo-random' numbers - but for almost all practical purposes they may be treated as random and 'unpredictable').

The function INT(X) produces the integer part of X - thus INT(2.7) is 2. The command LET X = INT(2*RND) assigns to X the value 0 or 1, with equal probability - c.f. coin tossing.

The following BASIC program simulates the rolling of a die, using the expression INT(6*RND) + 1

```
10  REM DICESIM
20  PRINT "INPUT NUMBER OF THROWS"
30  INPUT T
40  FOR I = 1 TO T
50  X = INT(6*RND) + 1
60  PRINT X
70  NEXT I
80  END
```

How Random Digit Tables are Produced

Pseudo-random numbers are produced by a formula, and so are not random. But if the formula is carefully chosen, the results are for practical purposes indistinguishable from those produced by a truly random process. Such random number generators are normally available on a computer, so that pseudo-random numbers can readily be used in programs.

We can illustrate the process as follows:

Take any 2 digit starting value x_0 and apply the formula $x_{n+1} = 2x_n + 1$ retaining only the last two digits (i.e. working modulo 100).

e.g. 1→ 3→ 7→ 15→ 31→ 63→ 27→ 55→ 11→ 23→ 47→ 95→ ...

After 40 steps the number 1 will reappear and the cycle is retraced. All these numbers are odd and they are not evenly distributed in the interval (0,100). The alternative formula $X_{n+1} = 3x_n + 1$ (mod 100), gives both odd and even numbers, but has a cycle length of only 20.

One generator actually used in computing is $X_{n+1} = 125xn + 1$ (mod M) i.e. multiples of M are subtracted to bring the result below M (M is chosen as $2^{14} = 16384$; powers of 2 are particularly easy to manipulate on a computer).

If you have a calculator, try tracing the cycles associated with $X_{n+1} = 2x_n + 1$ and $X_{n+1} = 3x_n + 1$, mentioned above, and investigate other such formulae.

Using a Computer to Generate Random Numbers

The computer program PSEUDR prints out sequences of pseudo-random numbers generated in this way, enabling one to investigate different generating rules. (see p 74).

The generating rule for the sequence is determined by line 80 e.g. to investigate $x_n = 2X_n + 1$, modulo 100, we replace line 80 by C = C * B + 1 and input the values S (=x_0), B (=2), and M (=100).

For $x_{n+1} = 125x_n + 1$, use S = 10, B = 125, M = 16384.

The effect of lines 60, 100-130 is to terminate the program when the sequence begins to repeat. (The sequence may not actually return to the initial value - for example if $x_{n+1} = 2x_n$, mod 1000, with $x_0 = 1$, the value 1 will never be reached again. Thus here we reset the value of S to x_5 for the test for repetitions, and in most circumstances this will avoid the difficulty).

Further Simulation Problems for Computer Solution

Most of these problems may be simulated using dice or random digits; some (nos. 12,16-19) are deterministic and may be simulated using step-by-step calculation (or in the case of no.17, by drawing).
The associated computer programs follow on pp. 74ff

1. GREENX

Simulate a pedestrian trying to cross the road, with traffic coming from the left, during a given time period, with probability L, and from the right with probability R. Pedestrian can only cross if there is no traffic from either direction. L and R are input. Output the number of time periods in which he is successful/unsuccessful in crossing.

2. DICE GAME

A die is rolled repeatedly until two '1's' and at least one other odd number are obtained (see p. 123). Simulate this game; output for each of N games (where N is input) the length of the game, and finally, the average game-length. Modify the program to simulate an alternative game (see p 129):-
A die is rolled repeatedly; if a '3' or '5' appears before a '1', stop at once. If a '1' appears before either a 3 or a 5, stop when at least two '1's' and one other odd number have appeared. Repeat 100 times. Find the average length of a game, and how many games end in each of the two possible ways. (DGAME)

3. FIV6 Problem

A die is rolled repeatedly until a target of at least two '5's' and at least three '6's' has been reached. What is the expected waiting time? Write a program to perform the experiment N times (where N is input) and print out the individual wait-time and the average for the N experiments. (Sample 114).
The problem may easily be generalised for a target of P '5's' and Q '6's', where P and Q are input, or modified to simulate the collection of random digits.

4. COLLN

A set of N cards is to be collected - at each stage the next card to be collected is equally likely to be any one of the set of N. (see p. 1).
Write a program to perform the experiment K times, where N and K are input, displaying the result of each experiment, and the average waiting time.
A generalisation is to collect A of the first card, B of the second card,

C of the third card etc., where the dimension of the vector (A,B,C,.....) and its components are input. (See COLLGEN - the program FIV6 is a special case of this, with target vector (0,0,0,0,P,Q). (p 78).

BEETLE simulates the game of Beetle - first a '6' must be collected (body), after which a '4' (tail) and four '3's (legs), and a '5' (head) must be obtained. Once the '5' has appeared, two '1's' and two '2's' (eyes and feelers) are needed. (p 79 and Appendix II).

Output the waiting time to achieve "Beetle".

5. SERV

Customers arrive in a shop where there is a single server. Those not being served form a queue. The (fixed) time to serve a single customer, T units, is input, the average time between arrivals, K units, is also input. (If K<T a large queue will develop). The total number of customers arriving, average waiting time per customer, total waste of server time, and maximum queue length, are of interest. (SERVSL gives a graphical display of detail). (pp 29, 30, 81).

6. TWOPUMP

A petrol station has two self-service pumps. The time to serve a single customer is fixed (input), the average time gap K between arrivals is also input. As customers arrive they go to a free pump; if no pump is free they go elsewhere. Output is the number of satisfied and unsatisfied customers at the end of a run. (pp 27, 81, 82).

NPUMP - generalise the above for a garage with N pumps, where N is to be input.

7. RUNS

Simulate the repeated tossing of a coin, and count the number of runs obtained (repeated occurrences of the same event). E.g. THHTTTHH contains runs of length 1,2,3,2).

Investigate the effect of using a biased coin (p≠0.5) and of lack of independence of events; (e.g. if the result is H, a fair coin is tossed. If the result is T, an unfair coin, (p(T)≠0.5), is tossed). (pp 13, 111).

8. WALK

Simulate the random walk of a particle on a line, either with absorbing barriers (pp 28, 85) or non-absorbing barriers. (BWALK, WALK). (p 86)

9. EHR

Simulate gas-diffusion using the Ehrenfest model (p 31) (e.g. use 1000 molecules initially in the left hand compartment, A). Either display the number in A at each stage, or output a graph (see EHR1) to show the progress of the diffusion. (*pp 96-7*).

10. HERM

Simulate the spread of infection (or a rumour) among a community of K individuals (where K is input). An actively infectious hermit A visits a randomly-chosen hermit B for help. If B has already had the disease, the epidemic dies out - otherwise A recovers and returns home, and B becomes actively infectious. Output is the total number of hermits infected. (HERM1 performs the experiment T times, where T is input). (*pp 7, 88-9*).

11. PINBALL

Write a program to simulate the operation of the pinball machine shown in the diagram (*ii*). Output the score and the game length for each of N experiments (where N is input), and the average score and game length. (*See p 26*).

12. DECAY2

Simulate the 2 stage decay of a chemical substance U→V→W (initially, U=1, V=W=0). Print out the amounts of U,V,W at intermediate stages and the maximum amount of the intermediate product V. (*pp 2, 93, 118*).

13. BROWNN

Simulate the two-dimensional Brownian motion of a particle which moves randomly one step in each time unit and with equal probabilities, p=¼, in any of the four directions N,S,E,W.
(Either print a list of the successive (X,Y) co-ordinates of the particle or (harder) display the output graphically in some way). (*pp. 93, 97*).

14. DICE

Simulate the tossing of 4 dice simultaneously. Obtain the frequency distribution of the sum of the scores on the four dice. (DICE4) (*pp 11, 92, 109*).

15. COINS

Simulate the tossing of N coins simultaneously where N(≤10) is input. Obtain the frequency distribution of the number which fall heads. (The Binomial distribution gives the theoretical probabilities). (NCOINS) (*pp 11, 92, 111*)

16. RUMOUR Problem (c.f. pp f, 3, and Engel (1977) p130)

In a town of (N+1) inhabitants a rumour is circulating. Everyone who hears the rumour continues to spread it, until he meets someone who has already heard it. Then he stops passing it on.

Subdivide the inhabitants into three groups X,Y,Z where -

X is the no. of people who have not heard the rumour

Y is the no. of people who are actively spreading it (infective)

Z is the no. of people who have ceased to spread it (immune)

Initially we have $X=N$, $Y=1$, $Z=0$ and throughout $X+Y+Z = N+1$

'Infection' $(X,Y,Z) \longrightarrow (X-1, Y+1, Z)$

'Immunisation' $\begin{cases} (X,Y,Z) \longrightarrow (X, Y-1, Z+1) \\ (X, Y, Z) \longrightarrow (X, Y-2, Z+2) \end{cases}$

} we consider these types of transformation

(the third of these being when two 'infectives' meet).

The rates of change of Y,Z will be proportional to the number of appropriate encounters. Thus in one unit of time -

$$X \longrightarrow X - P*X*Y$$

and $$Y \longrightarrow Y + P*X*Y - P*(Y*Z + \frac{2*Y*(Y-1)}{2}) = Y+P*X*Y - P*Y*(N-X)$$

where P is a constant.

The rumour dies away when $Y < 1$. (RUMOUR) p 96.

17. FOX AND HARE (c.f. Engel (1977), p130)

A hare starts from (0,0) and runs due North with speed P. A fox, initially at the point (x_0, y_0) chases him, running always directly towards the present position of the hare, with speed q. Find the 'curve of pursuit'. (FOX) p 96.

18. FOXES AND RABBITS

Consider an island which is thickly covered with grass and supports a large population of rabbits. Let the number of rabbits at a given moment be X. In the absence of external checks on population growth, X will increase exponentially ($\frac{dX}{dt} = AX$, where A is some constant, which leads to $X = X_0 e^{At}$). Suppose that a population of Y foxes also live on the island, feeding on the rabbits. If we assume no rabbits die of old age, the rate of decrease of rabbits will be proportional to XY - thus $-\frac{dX}{dt} = AX - BXY$

Similarly we assume that the rate of growth of the fox population is proportional to XY (more foxes, more cubs; more rabbits, more food) and since foxes <u>may</u> die of old age, we have $\frac{dY}{dt} = -CY + DXY$

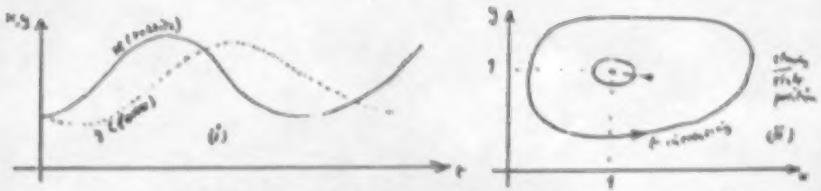
By suitably choosing the units of measurement for X and Y, we can arrange that A=B and C=D and the equations reduce to:-

$$\frac{dX}{dt} = AX(1 - Y), \quad \frac{dY}{dt} = CY(X - 1) \quad \ldots\ (1)$$

Clearly when X=1, Y=1, $\frac{dX}{dt} = \frac{dY}{dt} = 0$, and so X and Y will remain constant. (1,1) represents a steady state.

Suppose X=Y=0.5 then $\frac{dX}{dt} > 0$, and $\frac{dY}{dt} < 0$; the rabbits will increase and the foxes be reduced in number. In due course, when X exceeds 1, $\frac{dY}{dt} > 0$, and the foxes will begin to increase (and more rabbits will be eaten). When Y exceeds 1, $\frac{dX}{dt}$ goes negative and the rabbit population is rapidly reduced.

The behaviour may be represented by plotting x, y against t (in (i)):-



or by a graph in the X,Y plane, which can be shown to be a closed curve, (ii).

Note on the shape of the closed curves:-

$\frac{dy}{dx} = 0$ for $x = 1$, and $\frac{dy}{dx} = \infty$ for $y = 1$, so tangents are parallel to x and y axes respectively there.

If x and y both become very small (near(0,0)) the equations (1) are approximately

$\frac{dx}{dt} = ax$, $x = E_1 e^{at}$

$\frac{dy}{dt} = -cy$, $y = E_2 e^{-ct}$

} exponential growth and decay respectively

with $\frac{dy}{dx} = \frac{c}{a} \cdot \frac{Y}{X}$ from which $x^c y^a$ = constant

crossing $y = x$ in direction parallel ay + cx = 0.

For very large x and y, $\frac{dy}{dx} = \frac{-c}{a}$ and again tangents are parallel

to ay + cx = 0;

both x and y are changing rapidly with t (foxes multiplying and rabbits declining rapidly).

Write a simulation program using equations (1) and a step by step solution calculating the values of X and Y every H seconds:

(Take H = .01, and use the statements

X0 = X0 + A * X0 * (1 - Y0) * H

Y0 = Y0 + C * Y0 * (X0 -1) * H).

Print out the values of X0 and Y0 after every M^{th} step and continue until T pairs of coordinates have been printed. (Suggested M = 10, T = 60).

(Because of inaccuracies in the step by step solution, the calculated points may not quite form a closed curve - a more accurate step by step method is used in the program FOXRAB p. 95).

19. Flight of a Projectile with Air Resistance

The simple equations of motion of a particle moving under gravity are:

$$\frac{d^2y}{dt^2} = -g$$

$$\frac{d^2x}{dt^2} = 0$$

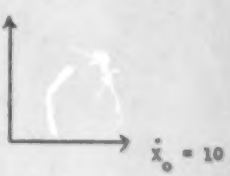
If air resistance proportional to velocity is introduced, in the opposite direction to that of motion, these become,

$$\frac{d^2y}{dt^2} = -g - kv\sin\alpha = -g - k\frac{dy}{dt}$$

$$\frac{d^2x}{dt^2} = -kv\cos\alpha = -k\frac{dx}{dt}$$

These equations <u>can</u> be integrated, but if, more realistically, the resistance is made proportional to v^2, <u>no</u> analytic solution is possible, and a step by step (or simulation) solution is the only possibility.

We consider, as a simple case, a particle projected with horizontal and vertical components of 10m s^{-1} and suppose resistance $\propto (vel)^2$, K=0.003



Consider time steps of $\frac{1}{10}$ sec. (distances in m. speeds in ms^{-1}, accelerations in ms^{-2}; take $g = 9.8\ ms^{-2}$).

Initially X=0, Y=0, $\dot{X}$ =10, $\dot{Y}$ =10

$$\ddot{Y} = -9.8 - 0.003 \times (10^2+10^2) \times \sin 45^\circ$$

$$\ddot{X} = -0.003 \times (200) \times \cos 45^\circ$$

After $\frac{1}{10}$ sec (using the values of X , Y at the beginning of the motion) we have approximately:-

$$X = 10 \times \tfrac{1}{10} = 1, \qquad Y = 10 \times \tfrac{1}{10} = 1$$

and also $\dot{X} = 10 - (0.003 \times 200 \cos 45^\circ)\frac{1}{10} = 10 - 0.0426 = 9.9576$

$$\dot{Y} = 10 - (9.8 + 0.003 \times 200 \tfrac{1}{\sqrt{2}})\tfrac{1}{10} = 10 - 0.98 - 0.0426 = 8.9774$$

Now, using these values of $\dot{X}$ and $\dot{Y}$ we can calculate the direction of motion and the speed:-

$V = \sqrt{\dot{X}^2 + \dot{Y}^2} = \sqrt{179.748} = 13.407$

$\sin\alpha = \frac{\dot{Y}}{V} = 0.66961 \quad \cos\alpha = \frac{\dot{X}}{V} = 0.74272$

In the general case we may write $\begin{cases} \ddot{Y} = -9.8 - 0.003.V.\dot{Y} \\ \ddot{X} = -0.003.V.\dot{X} \end{cases}$

(so there is no need to calculate $\sin\alpha$ and $\cos\alpha$)

Using the new values of $V, \dot{X}, \dot{Y}, \ddot{X}, \ddot{Y}$ we can calculate values after $\frac{2}{10}$ sec:-

$X = 1 + 9.9576 \times \frac{1}{10} = 1.9958$

$Y = 1 + 8.9774 \times \frac{1}{10} = 1.8978$

$\dot{X} = 9.9576 - 0.003 \times 13.407 \times 9.9576 \times \frac{1}{10} = 9.9175$

$\dot{Y} = 8.9774 - (9.8 + 0.003 \times 13.407 \times 8.9774)\frac{1}{10} = 7.9615$

and so on.

A computer program to calculate these results is given on p 77.

This uses a more accurate step by step method. Results for this and for the simple method used above are given in the table which follows. (It may be seen that reducing the step length - ie. calculating $X, Y, \dot{X}, \dot{Y}$ at shorter intervals of (say) $\frac{1}{100}$ sec has no effect in the more accurate program).

	Simple version as above				More accurate step-by-step method	
Step length	Δt=0.1		Δt=0.01		Δt=0.1	
Time (sec)	X	Y	X	Y	X	Y
0.1	1.000	1.000	0.998	0.954	0.998	0.949
0.2	1.996	1.898	1.992	1.806	1.992	1.796
0.3	2.988	2.694	2.982	2.558	2.982	2.543
0.4	3.975	3.389	3.969	3.208	3.969	3.189
0.5	4.960	3.984	4.952	3.759	4.952	3.734
2.0	19.362	1.090	19.342	0.242	19.339	0.148
(2.01)					19.526	-0.046
(2.03)			19.621	-0.050		
2.1	20.297	0.125			20.273	-0.860
2.2	21.229	-0.934			(results for t=0.01 are virtually identical)	

20. Birthdays

Suppose a group of N people are assembled in a room. What is the chance that at least 2 of them have their birthday on the same day?

Write a program to simulate the situation, and determine how the probability varies as N increases.

(Use an array DIM A(365) - if someone has a birthday on the I^{th} day of the year, let A(I) = 1. Start with A(I) = 0 for every I. Ignore leap years at first).

A SIMPLE INTRODUCTION TO SIMULATION

Part II

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INSTITUTE OF EDUCATION

UNIVERSITY OF KEELE 75 July 1980

Section 6 Computer Program Listings

This section contains annotated listings of the computer programs referred to elsewhere in the text. An index list of these is given below, followed by some notes which may be useful if they are to be adapted for use on other computer systems.

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Notes on the GEC 4082 BASIC Programs

The programs which follow make use of some features of 4082 BASIC which are non-standard, though many of them will be available in somewhat similar form on other computers. In most cases, any modifications needed will be minor, but for convenience these features are listed below.

LET The word LET is optional - and is not used in these programs.

END may also be omitted.

Initial values. All variables undefined by the programmer are given the value 0 (if numerical), or " " (i.e. 'nil'), for string variables .
Variables are usually explicitly initialised in these programs.

Default sizes for arrays. Arrays which are not explicitly dimensioned by the programmer are given a default size of 10 i.e. A(0), A(1),.....A(10)
Again an attempt has been made here to ensure that all arrays are explicitly dimensioned.

Output. The usual conventions for PRINT statements apply - in particular ⓒ gives adjacent and ⓘ gives spaced following output.
PRINT by itself gives a new line.
(Since output is to a buffer, it is sometimes necessary to clear the buffer by inserting an extra PRINT statement at the end of a program, to ensure that all output is printed. This explains the presence of the final PRINT statement in WELLS1, for example).

FILES statements.
Output is normally to the VDU but can be directed to a file, .FILE1, by the following:-

```
10   FILES#1, ".FILE1"
100  PRINT#1, X,Y        (Values of X and Y are printed to .FILE1)
```

(Names of all 4082 permanent files begin with a full stop e.g. .FILE2).
The symbol ⓥ is used to indicate the VDU, so that:

```
10   FILES#1, "*"
100  PRINT#1,X,Y
```

produces exactly the effect of

```
10   (blank)
100  PRINT X,Y
```

Thus, in conversion, FILES statements should be omitted, and #1, should be omitted from PRINT statements.

(So why bother?? By altering just the statement 10, output can be redirected to a specific file and stored there, without the need to amend every PRINT statement in the program).

Random numbers.
X = RND assigns a random number $0 \leq X < 1$ to X
RANDOMISE selects an unpredictable starting value - so that successive runs of a program do not give identical values of RND.

Multi-statement Lines

Several statements may appear on one line, separated by ①

This is useful in dealing with conditional statements:-

e.g. IF A(J) = 0 THEN S = S - 1 : GOTO 190 (see EHR1)

The general form is IF ⟨condition⟩ THEN ⟨statement 1⟩ : ⟨statement 2⟩ : S3:.....

If the condition is true, all the statements are executed;

if it is false, none of them are.

Such a statement can be replaced as follows:-

```
100  IF ⟨not-condition⟩ THEN 150
110  ⟨statement 1⟩
120  ⟨statement 2⟩
130  ⟨statement 3⟩
       etc
150  ⟨continue⟩
```

Note 'Multi' brackets ① and ② are used here to enclose that which is replaced

FOR loops. Closing NEXT statements can be amalgamated:

Thus NEXT J
NEXT I

may be replaced by NEXT J,I

PRINT USING. Output can be produced in a specific format.

Thus PRINT USING "1#.2#",X,Y

prints the values of X and Y (on the same line without any spaces between), each being given with 1 digit before and two after the decimal point.

(On some systems this may be achieved by an instruction of the form PRINT INT(100*X)/100)

Matrices can be initialised by a single command,

e.g. MAT X = ZER (N)

which has the effect of

```
FOR I = 0 to N
X(I) = 0
NEXT I
```

while MAT X = ZER (N,N) replaces

```
FOR I = 0 to N
FOR J = 0 to N
X(I,J) = 0
NEXT J
NEXT I
```

(similarly MAT X = IDN(N) produces a (square) identity matrix)

Character strings may be used to represent numbers :-

The instruction X$ = CHR$ (48 + I) converts the number representation of I to the string representation.

(This is used in the program FOXRG, p.96, where results are displayed by output of string array, most of the elements being " " - i.e. 2 spaces)

PSEUDR

```
0010 FILES #1,"o"
0020 REM PSEUDO RANDOM - LINE 80
0030 PRINT #1,"INPUT START,BASE,MODULUS"
0040 INPUT S,B,M
0050 C=S
0060 I=1
0070 PRINT #1,C;
0080 C=C*B
0090 C=C-M*INT(C/M)
0100 IF C=S THEN 170
0110 IF I>5 THEN 70
0120 I=I+1
0130 S=C
0140 REM SETS C= FIFTH MEMBER OF SEQUENCE
0150 REM IN CASE NOT IMMEDIATELY CYCLIC
0160 GOTO 70
0170 PRINT #1,C;
0180 PRINT #1,
0190 END
```

Variables
C = integer output
S = initial value of C (after 5 C's = 0.140)
B = multiplying factor (1.80)
M = modulus; C is reduced modulo M
I = counter for cycles of C values

GREENX

Simulation of pedestrian crossing busy traffic.

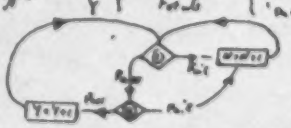
```
0010 REM CROSSING THE ROAD
0020 RANDOMISE
0030 PRINT"INPUT PROB OF VEHICLE ARRIVING FROM L"
0040 INPUT L
0050 PRINT L
0060 PRINT "INPUT PROB OF VEHICLE ARRIVING FROM R"
0070 INPUT R
0080 PRINT R
0090 PRINT "INPUT NUMBER OF TIME PERIODS"
0100 INPUT K
0110 PRINT K
0120 N=0
0130 Y=0
0140 FOR I=1 TO K
0150 IF RND>L THEN 180
0160 N=N+1
0170 GOTO 220
0180 IF RND>R THEN 210
0190 N=N+1
0200 GOTO 220
0210 Y=Y+1
0220 NEXT I
0230 PRINT
0240 PRINT
0250 PRINT "NUMBER OF TIME PERIODS ";K
0260 PRINT "VALUES OF Y,N,Y/K ";Y;N;Y/K
0270 PRINT "FOR PROBABILITIES L & R  ";L;R
```

~ left traffic

~ right traffic

cross

Variables:-
L, R prob. of arrival from left, right
K no. of trials - (control variable I.)
N, Y accumulated totals 'fail to cross' 'across fail'



DGAME2

Simulation of game: to throw (3 or 5) or (if '1' appears first) two '1's and (3 or 5).

```
0010 REM DICE GAME II
0020 REM  (3 OR 5)  **OR** TWO 1'S-AND-(3 OR 5)
0030 RANDOMISE
0040 C0=0
0050 C1=0
0060 I1=0
0070 N=20
0080 FOR Z=1 TO N
0090 C=1
0100 A=2
0110 B=1
0120 I=0
0130 R=RND
0140 I=I+1
0150 REM TEST FOR 'EVEN'
0160 IF R>0.5 THEN 130
0170 REM TEST FOR '1'
0180 IF R<1/6 THEN 220
0190 B=B-1
0200 IF C=1 THEN C0=C0+1:GOTO 270
0210 GOTO 240
0220 A=A-1:C=0
0230 REM '1' THROWN
0240 IF B>0 THEN 130
0250 IF A>0 THEN 130
0260 C1=C1+1
0270 PRINT I;
0280 I1=I1+I
0290 NEXT Z
0300 PRINT "AVERAGE GAME LENGTH = ";I1/N
0310 PRINT "ABSORBED IN 'ODD' ";C0
0320 PRINT "ABSORBED IN 'ONE' ";C1
0330 END
```

initialise

initialise

'3' or '5'

game finishes via 'odd' (3 or 5)

'1' thrown

game finishes via '1'

Variables:-

N = no. of expts. (control variable Z)
I1 = cumulative game length
A = no. of '1's required
B = no. of (3 or 5)'s required
I = no. of steps in current game
R = random number $0 < R < 1$.

C set at 1 unless a '1' is thrown first (l. 220)

In l. 200, C=1 => game concludes
but, if C=0, carry on as in basic (I).

DGAME

Simulation of game: throw two '1's and either 3 or 5.

```
10 REM DICE GAME I  TWO 1'S AND (3 OR 5)
20 RANDOMISE
30 I1=0
40 N=20
50 FOR Z=1 TO N
60 A=2
70 B=1
80 I=0
90 R=RND
00 I=I+1
10 REM TEST FOR 'EVEN'
20 IF R>0.5 THEN 90
30 REM TEST FOR '1'
40 IF R<1/6 THEN 170
50 B=B-1        ('3' or '5')
60 GOTO 190
70 A=A-1
80 REM '1' THROWN
90 IF B>0 THEN 90
00 IF A>0 THEN 90
10 PRINT I;
20 I1=I1+I
30 NEXT Z
40 PRINT "AVERAGE GAME LENGTH = ";I1/N
50 END
```

'even' thrown

i.e. if '1' thrown

Variables:-

N = no. of expts. (Control variable Z)

I1 = cumulative game length

A = no. of '1's required

B = no. of (3 or 5)'s required

I = no of steps in current game

R = random number $0 \le R < 1$.

FIV6

```
0010 REM TO COLLECT P 5'S  AND  Q  6'S
0020 REM BY ROLLING A DIE
0030 RANDOMISE
0040 PRINT "INPUT P,Q"
0050 INPUT P,Q
0060 K=6
0070 PRINT "HOW MANY REPETITIONS?"
0080 INPUT N
0090 J=0          initialise cumulative wait.
0100 PRINT "WAITTIMES"
0110 PRINT
0120 REM BEGIN  N  EXPTS
0130 FOR Z=1 TO N
0140 I=0
0150 A=0
0160 B=0
0170 I=I+1
0180 R=RND
0190 IF K*R<2 THEN230      '5' or '6'
0200 REM NEITHER 5 NOR 6
0210 GOTO 170
0220 REM TEST FOR '5'
0230 IF K*R<1 THEN A=A+1:GOTO 270
0240 REM '6' OBTAINED
0250 B=B+1
0260 REM TEST WHETHER TARGET REACHED
0270 IF A<P THEN 170
0280 IF B<Q THEN 170
0290 PRINT I;      wait time for (P,Q)
0300 J=J+I         cumulative wait.
0310 NEXT Z
0320 PRINT
0330 PRINT "AVGE WAIT =";J/N
0340 END
```

wait no. of '5's, '6's

Variables:-

P,Q. P '5's and Q '6's are required

K $\frac{1}{\text{prob of each outcome}}$ eg. K=6, $P=\frac{1}{6}$,

N no. of expts

J cumulative wait ($=\Sigma I$)

Z control variable for expts

I count of trials. 0 = wait, I = wait time

A count no of '5's

B count no of '6's

R random number, $0 \le R < 1$, to decide outcome

COLLN

```
0010 REM SIMULATES COLLECTION OF SET OF N CARDS
0020 PRINT "INPUT NUMBER OF CARDS IN SET TO BE COLLECTED"
0030 INPUT N
0040 PRINT N
0050 PRINT "INPUT NUMBER OF TRIALS"
0060 INPUT K
0070 PRINT K
0080 DIM A(100)
0090 T=0                    initialise cumulative total of C values
0100 PRINT "NUMBER COLLECTED TO ENSURE COMPLETE SET"
0110 FOR Z=1 TO K
0120 REM NEW EXPT : RESET TO EMPTY
0130 FOR I=1 TO N
0140 A(I)=0                 new card A(I)
0150 NEXT I
0160 REM S=NO REMAINING TO BE COLECTED
0170 REM C=COUNT OF CARDS RECEIVED
0180 S=N                    number remaining to be collected
0190 C=0                    count of no. of cards
0200 J=INT(RND*N+1)         choose next card
0210 IF A(J)=0 THEN S=S-1   'new' card
0220 REM S ONLY ALTERED IF 'NEW' CARD
0230 A(J)=A(J)+1            mark card J as 'processed'
0240 C=C+1                  no. of cards
0250 REM FULL SET?
0260 IF S=0 THEN 280        yes
0270 GOTO 200               no
0280 PRINT C;
0290 REM WAITING TIME
0300 T=T+C                  augment total
0310 NEXT Z
0320 PRINT
0330 PRINT "AVERAGE = ";T/K
0340 REM FOR K EXPTS
0350 END
```

Variables:-

N = no. of cards in set, control variable J
K = no. of trials, control variable Z.
A(I) no. of cards of type I collected (here a type)
C no. of cards needed to ensure complete set collected
T cumulative total of C values
S no. of cards remaining to be collected.

COLLEEN

Simulation of general collating function. - Pair 29.

```
0010 V=0          set overflow count to zero
0020 REM TARGET VECTOR (P,Q,R,S...) MEANS P OF "A",Q OF "B", ETC
0030 RANDOMISE
0040 DIM F(200)
0050 PRINT "INPUT DIMENSIONS OF TARGET VECTOR"          N ≤ 25 - see Note
0060 INPUT K                                                 130, 150, 160.
0070 PRINT K
0080 PRINT "HOW MANY REPETITIONS?"
0090 INPUT N
0100 PRINT N
0110 I1=0         number of game cycles
0120 PRINT "INPUT TARGET VECTOR"
0130 MATINPUT A(K)
0140 MATPRINT A ;
0150 DIM B(20)
0160 MAT B=A                    copy of A for use, Pair 130.
0170 PRINT "WAITTIMES"
0180 PRINT
0190 FOR S=1 TO N
0200 I=0
0210
0220
0230 I=I+1
0240
0250 R=INT (K*RND+1)      choose positions from {1, 2, ... K}
0260
0270 A(R)=A(R)-1
0280 FOR L=1 TO K:IF A(L)>0 THEN 230:NEXT L      check whether all A(I) reduced to zero
0290 PRINT I;
0300 IF I>100 THEN I=100:V=1      Reduce I to a valid subscript on overflow,
0310 F(I)=F(I)+1      freq. classification      & set overflow marker.
0320 I1=I1+I      cumulative sum of waits
0330 MAT A=B      re-initialise A via copy B.
0340 NEXT S
0350 PRINT
0360 PRINT "FREQUENCIES"
0370 MATPRINT F;      game cycles.                    } output
0380 IF V=1 THEN PRINT "*** OVERFLOW HAS OCCURRED ***"
0390 PRINT
0400 PRINT "AVGE WAIT =";I1/N                        ]
0410 END
```

Variables

V set to 0 ; if overflow occurs, set to 1

F freq. classification of game cycles

K (control variable 1) dimension of target vector

N (control variable 2) no. of games

I1 cumulative sum of game cycles

A( ) target vector

B( ) copy of ....

I game cycle (in wait)

R random digit from {1, 2, ... K}

BEETLE
Simulation of the game.

```
0010 REM TARGET VECTOR (P,Q,R,S,...) MEANS P OF "A", Q OF "B", ETC
0020 RANDOMISE
0030 K=6
0040 PRINT "HOW MANY REPETITIONS?"
0050 INPUT N
0060 PRINT N
0070 PRINT "DETAIL OF MOVES REQUIRED AT EACH STAGE?:YES/NO"
0080 INPUT A$
0090 J=0
0100 FOR B= 1 TO N
0110 RESTORE
0120 I1=0
0130 GOSUB 320
0140 GOSUB 320
0150 PRINT "BEETLE IN ";I1;" MOVES"
0160 J=J+I1
0170 PRINT
0180 NEXT B
0190 PRINT
0200 PRINT "AVERAGE GAME LENGTH WAS ";J/N
0210 END
0220 MAT READ A(K)
0230 I=0
0240 I=I+1
0250 R=INT(K*RND+1)
0260 A(R)=A(R)-1
0270 IF A$<>"NO" THEN MATPRINT A;
0280 IF A(5)=0 THEN A(5)=-1:A(1)=2:A(2)=2
0290 FOR L=1 TO K: IF A(L)>0 THEN 240:NEXT L
0300 PRINT I;"";
0310 I1=I1+I
0320 RETURN
0330 DATA 0,0,0,0,-1,1
0340 DATA 0,0,4,1,1,0
```

Length of game

to get 'E'

& get S.(2',1'), 4, 3^4

number of games

If yes, next state is printed

number in game length

for one game (1000, 760)

to get 'E' (body) : 5th position not =1 to 0 280 not activated (but 0 290 can be)

to get 4 legs, tail, head.

number of moves to end of stage

game length

display array

check stage complete?

state vector

If game obtained, test 2^5 pattern

reset } end game
count

Length

'Moves'

Key:

Set tests for body, head, tail, 4 legs, 2 legs, 2 'feelers'.
The body must be obtained first ('Moves' = 'E')
When the head is obtained ('5'), collection of legs & feelers can begin ('>1').

N = no. of games (control switch 2)
K = dimension of target vector (= 6, this)
J = cumulated sum of (total) game length.
I1 = } length of { game
I = } { stage
A$ = NO => detailed state of game not displayed.
A(I) = state of each target component.

SERVE

Simulation of single-server queue, with graphical output of detail

```
0010 S=0          server idle time
0020 DIMQ$40      (at most big this is 40 characters for 'picture' of Q.)
0030 C=0          no. of customers
0040 RANDOMISE
0050 Q1=0         Σ queue lengths
0060 Q=0          current queue
0070 PRINT "INPUT  SERVE TIME"
0080 INPUT T
0090 PRINT "T= ";T
0100 PRINT "INPUT AVERAGE GAP BETWEEN ARRIVALS"
0110 INPUT K
0120 PRINT "K=";K
0130 PRINT "DETAILS IN SLOW MOTION REQUIRED? INPUT YES/NO"
0140 INPUT D$
0150 PRINT D$
0160 IF D$="NO" THEN 200
0170 PRINT "PICTURE AT  **END**  OF EACH INTERVAL"
0180 PRINT "ARRIVAL";TAB(10);"SERVER";TAB(30);"QUEUE"
0190
0200 P1=T              clock reads 0 since ('end of previous interval')
0210 FOR I =1 TO 100
0220 C$=" "            no. yet arrived picks 0 car
0230 P1=P1+1           minute later
0240 IF K*RND<1 THEN C=C+1:C$="*":GOTO 290     arrival, increase C and Q$
0250 IF P1<T THEN 310                          no arrival, some completed
0260 IF Q>0 THEN Q=Q-1:P1=1:GOTO 310           queue reduced, new server free
0270 S=S+1                                     server free and no queue - idle started
0280 GOTO 310
0290 IF P1>T THEN P1=1 :GOTO 310               server free - later customer
0300 Q=Q+1                                     customer joins queue
0310 Q1=Q1+Q                                   total waiting time
0320 IF Q>M THEN M=Q                           queue increased
0330 IF D$="NO" THEN 400          no 'picture'
0340 PRINT C$;TAB(10);
0350 Q$=""
0360 FOR J=1 TO Q:Q$=Q$+"*":NEXT J             build up picture of situation
0370 IF P1>T THEN PRINT " F":GOTO 390          server switch to F('free') at end of service time (T MINUTES)
0380 PRINT P1;
0390 PRINT TAB (20);Q$
0400 NEXT I
0410 PRINT
0420 PRINT
0430 PRINT M,"=MAX. QUEUE"                     finishing output
0440 PRINT "FINAL QUEUE = ";Q
0450 PRINT Q1/100 ,"=AVERAGE WAITING TIME"
0460 PRINT C,"TOTAL NO OF CUSTOMERS"
0470 PRINT S,"TOTAL WASTE OF SERVER TIME"
0480 END
```

Variables: (so far)

85

```
D$   Yes/no for detail
C$   picture of customer arrival
Q$   picture of queue.
I    control variable for loop of Q$.
```

```
0010 B=0                    server idle time
0020 C=0                    no. of customers
0030 RANDOMISE
0040 Q1=0                   X queue length
0050 Q=0                    count queue
0060 PRINT "INPUT  SERVE TIME"
0070 INPUT T
0080 PRINT "INPUT ARRIVAL PROB"
0090 INPUT K1
0100 K=1/K1                 K = average gap between arrivals
0110 P1=T                   set server at 'ready'
0120 FOR I =1 TO 100        repetition
0130 P1=P1-1                advance server condition
0140 IF K*RND<1 THEN C=C+1:GOTO 190     customer arrives
0150 IF P1<T THEN 210       no customer, but server busy
0160 IF Q>0 THEN Q=Q-1:P1=1:GOTO 210    server free: reduce Q by 1, and serve
0170 B=B+1                  server free and no queue - time wasted
0180 GOTO 210
0190 IF P1>T THEN P1=1 :GOTO 210        allocate customer to server
0200 Q=Q+1                  increase queue size
0210 Q1=Q1+Q                add queue length
0220 IF Q>R THEN R=Q        store max. length of queue
0230 NEXT I
0240 PRINT R,"=MAX QUEUE"
0250 PRINT Q1/100 ,"=AVERAGE WAITING TIME"
0260 PRINT C,"TOTAL NO OF CUSTOMERS"
0270 PRINT B,"TOTAL WASTE OF SERVER TIME"
```

Variables:
S - total server idle time
C - total number of customers
Q1 - cumulative length of queue
R - record queue
T - service time
K1 = probability of arrival in unit time
K = $\frac{1}{K1}$ - average unit between arrivals
P1 - state of server. P1=T at the start
Q - max size of Q
I - control variable for number of time units (customers + time)

Transport
Petrol pump simulation - 2 pumps

```
0005 RANDOMISE
0010 B=0:U=0:I=0
0015 INPUT V,K              service time, average gap between vehicles
0020 D=0:T=0                set arrivals & empty
0030 B=B+1:T=T+1:I=I+1      advance time unit count and pump availability
0040 IF I=100 THEN 100
0050 IF K*RND<1 THEN 70
0060 GOTO 30
0070 IF D>V THEN D=1:B=B+1:GOTO 30      assign customer to {X / Y}
0075 IF T>V THEN T=1:B=B+1:GOTO 30
0080 U=U+1                  no free pump ... customer lost.
0090 GOTO 30
0100 PRINT "U=",U           } end put
0110 PRINT "B=",B
```

Variables:
S } customer served
U } customer unserved
I count of time units
V service time in units
K average gap between vehicles
X } status of pumps { 1 - free if X > V
Y } { 2 - free if Y > V

NIJMP

Gump simulation

```
0010 REM PETROL PUMPS
0020 PRINT "HOW MANY PUMPS?"
0030 INPUT N
0040 RANDOMISE
0050 PRINT "GIVE VALUES OF SERVICE TIME AND GAP-TIME"
0060 REM K = AVERAGE GAP BETWEEN VEHICLES
0070 INPUT V,K
0080 PRINT V;K
0090 MAT P=ZER(N)          for pump status. (This statement dimensions P)
0100 REM 5 EXPTS
0110 FOR J=1 TO 5
0120 REM PUMP STATUS: P(I)>V => FREE
0130 FOR I=1 TO N
0140 P(I)=V                set all pumps at 'ready'
0150 NEXT I
0160 REM SAT/UNSAT CUSTOMERS
0170 S=0:U=0               initialise
0180 REM 100 TIME INTERVALS
0190 FOR C=1 TO 100
0200 FOR I=1 TO N
0210 P(I)=P(I)+1           advance availability of each pump by 1 - if P(I)=V, then I'th pump is free.
0220 NEXT I
0230 IF K*RND<1 THEN 280
0240 REM NO ARRIVAL
0250 GOTO 310 ------------------------------------------------------>
0260 FOR I=1 TO N          arrival
0270 REM ASSIGN CUSTOMER TO FREE PUMP?
0280 IF P(I)>V THEN P(I)=1:S=S+1:GOTO 310    customer satisfied
0290 NEXT I
0300 U=U+1    no free pump : customer lost
0310 NEXT C                <-------------------------------------
0320 REM RESULTS OF EXPT : UNSATISFIED/SAT
0330 PRINT "U=",U          results for single expt.
0340 PRINT "S=",S
0350 NEXT J
0360 END
```

Variables

N number of available pumps
P(I) availability of pump I (available if P(I) = V)
V Pump is service a vehicle (is 'busy')
K average gap between vehicles (prob(arrival) = $\frac{1}{K}$)
I repetitions of expt.
I control variable for pumps
S customers served, satisfied
U customers unserved, unsatisfied
C control variable for time units

RUNS1

```
0010 DIM H(100)
0020 DIM B(100)
0030 V=0
0040 REM SIMULATION OF COIN TOSSING - RUNS
0050 RANDOMISE
0060 FILES 01,"*"
0070 PRINT 01,"INPUT NUMBER OF THROWS"
0080 INPUT T
0090 PRINT T
0100 MAT B=ZER(T)
0110 PRINT 01,"INPUT PROBS P,Q"
0120 REM                    2ND
0130 REM              ( P       1-P)
0140 REM         1ST  (             )
0150 REM              ( 1-Q      Q)
0160 INPUT P,Q
0170 PRINT P,Q
0180 REM P=Q=0.5 FOR UNBIASED COIN,P=1-Q<>0.5 FOR BIAS
0190 REM P<>1-Q FOR DEPENDENCE
0200 L=2
0210 R=0
0220 Y=INT(2*RND+1)
0230 REM ARBITRARY CHOICE OF "LAST" TRIAL
0240 REM WHICH ***1*** INCLUDED WHEN COUNTING RUN LENGTHS
0250 PRINT
0260 PRINT "RUN LENGTHS"
0270 PRINT
0280 FOR I=1 TO T
0290 R=RND
0300 L=L+1
0310 IF L>99 THEN R=L
0320 IF V=2 THEN 370
0330 REM  CHOOSE FROM  1ST RUN
0340 IF R<P THEN X=1 :GOTO 390
0350 X=2 :GOTO 390
0360 REM CHOOSE FROM 2ND RUN
0370 IF R<Q THEN X=2:GOTO 390
0380 X=1
0390 B(I)=X
0400 IF X=Y THEN 440
0410 REM  CHANGE ----  NEW RUN
0420 PRINT L;
0430 H(L)=H(L)+1:L=0:Y=X
0440 NEXT I
0450 REM LAST (PARTIAL) RUN
0460 L=L+1: PRINT L;H(L)=H(L)+1
0470 IF L>M THEN R=L
0480 PRINT
0490 PRINT "RECORD OF STATES"
0500 MATPRINT B;:PRINT
0510 GOSUB 530
0520 END
0530 REM HISTOGRAM SUBROUTINE
0540 PRINT "HISTOGRAM OF RESULTS"
0550 PRINT
0560 T1=0:T2=0
0570 T0=0
0580 FOR J=1 TO R
0590 T0=T0+H(J)
0600 T1=T1+H(J)*J
0610 T2=T2+H(J)*J^2
0620 PRINT J;TAB(12);
0630
0640 FOR B=1 TO H(J):PRINT "*";:NEXT B
0650 PRINT
0660 NEXT J
0670 PRINT
0680 IF V=1 THEN PRINT "*** NB OVERFLOW HAS OCCURRED ***":PRINT
0690 PRINT "MEAN = ";T1/T0
0700 PRINT "S.D. = ";SQR(T2/T0-T1^2/T0/T0)
0710 RETURN
```

Variables

H() long storage for length of runs
B() record of states "0" (=0) or "1" (=1)
T no. of throws
P prob of A followed by A
Q prob of B followed by B
L length of current run
M max. length of run
Y, X outcome of last, current trial
V coupling indicator, set at 1 (Ind)

Note

If it is likely that a run length will exceed 99, the coupling indicator V is set likely to be needed.
Additional program lines will be added:
625 IF L>99 then L=99:V=1
655 IF L>99 then L=99:V=1
665 IF M>99 then M=99:M=99:V=1

Variables

T0, T1, T2
If T0, T1, T2 of
I, Z ... working

RUNS

```
0010 REM SIMULATION OF COIN TOSSING - RUNS
0020 RANDOMISE
0030 FILES @1,"o"
0040 PRINT @1,"INPUT NUMBER OF THROWS"
0050 INPUT T
0060 PRINT @1,"INPUT PROBS P,Q"
0070 REM                      2ND
0080 REM                ( P       1-P)
0090 REM           1ST  (               )
0100 REM                ( 1-Q     Q)
0110 INPUT P,Q
0120 REM P=Q=0.5 FOR UNBIASED COIN,P=1-Q<>0.5 FOR BIAS
0130 REM P<>1-Q FOR DEPENDENCE
0140 C=0
0150 Y=INT(2*RND+1)
0160 REM ARBITRARY CHOICE OF "LAST" TRIAL
0170 FOR I=1 TO T
0180 R=RND
0190 IF Y=2 THEN 240
0200 REM  CHOOSE FROM  1ST ROW
0210 IF R<P THEN X=1 :GOTO 260
0220 X=2 :GOTO 260
0230 REM CHOOSE FROM 2ND ROW
0240 IF R<Q THEN X=2:GOTO 260
0250 X=1
0260 PRINT @1,X;
0270 IF X<>Y THEN C=C+1
0280 REM  CHANGE  ----  NEW RUN
0290 Y=X    update
0300 NEXT I
0310 PRINT @1,"NUMBER OF RUNS =";C
0320 PRINT @1,"AVERAGE RUN LENGTH = ";T/C
0330 END
```

Note - this is input to the RUNS, + only counts the number of runs in each sept.

1st no of which

Variables

P prob. '1' follows '1'
Q prob. '2' followed by '2'
C count of number of runs
Y outcome of 'last' trial
X outcome of current trial
R random no.

Output

RWALK

```
0010 REM RANDOM WALK ON LINE (DRUNKARD)
0020 RANDOMISE
0030 PRINT "INPUT NO OF TRIALS"
0040 INPUT K:PRINT K
0050 PRINT "INPUT PROB OF MOVEMENT TO LEFT"
0060 INPUT P:PRINT P
0070 H=0:S=0
0080 Q=0
0090 FOR J=1 TO K
0100 REM X = CURRENT POSITION AFTER T STEPS
0110 X=0
0120 FOR T=1 TO 100
0130 IF RND>P THEN X=X+1:GOTO 150
0140 X=X-1
0150 IF X>3 THEN H=H+1:GOTO 180
0160 IF X<-3 THEN S=S+1:GOTO180
0170 NEXT T
0180 PRINT T;
0190 Q=Q+T
0200 NEXT J
0210 PRINT
0220 PRINT "H= ";H;" S= ";S
0230 PRINT "AVERAGE WALK WAS ";Q/K
0240 PRINT
0250 END
```

Variables

H = no. of times walker { Home
S = { Sober
Q = cumulative sum of walk lengths
K = no. of trials, control variable J
P = prob. of step to left (Random Number)
X = walk position
T = control variable for count no. of steps

Notes

(a) L150 accounts for walk still counted NO HOME. Loop is left at end of walk

(b) By replacing '3' in L150, 160 by 5, could show the dependence of average walk on $\sqrt{5}$

WALK

(on chasing random walk, investigating $d = c\sqrt{n}$ — see p129)

```
 0010 REM RANDOM WALK ON LINE
 0020 RANDOMISE
 0030 PRINT "INPUT NO OF TRIALS"
 0040 INPUT K:PRINT K
 0050 PRINT "INPUT PROB OF MOVEMENT TO LEFT"
 0060 INPUT P:PRINT P
 0070 PRINT "DO YOU WANT DETAIL DISPLAY?  YES/NO "
 0080 IN.  T A$:PRINT A$
 0090 REM  A(T) = CURRENT SUM OF ABSOLUTE DISTANCES FROM 0
 0100 REM  AT TIME T
 0110 DIM A(100)
 0120 FOR J=1 TO K
 0130 REM X = CURRENT POSITION AFTER T STEPS
 0140 X=0
 0150 IF A$="NO" THEN 180
 0160 PRINT "SIMULATION NUMBER ";J      heading
 0170 PRINT TAB(30);"!"                 centre line
 0180 FOR T=1 TO 100
 0190 A(T)=ABS(X)+A(T)                  sum of absolute distances from 0
 0200 IF RND>P THEN X=X+1:GOTO 220      move R
 0210 X=X-1                             move L
 0220 IF A$="NO" THEN 280               no display
 0230 REM DISPLAY
 0240 IF ABS(X)>30 THEN PRINT "OUTSIDE RANGE":GOTO 510 ---> (1)   If ABS(X) too large, print message, & abandon.
 0250 IF X>0 THEN PRINT TAB(30);"!";TAB(30+X);"*"
 0260 IF X=0 THEN PRINT TAB(30);"!"
 0270 IF X<0 THEN PRINT TAB(30+X);"*";TAB(30);"!"      } appropriate choice of detail marking
 0280 NEXT T,J
 0290 IF A$="NO" THEN 310
 0300 PRINT "DISPLAY ENDS ******"
 0310 Z=0                               initialise sum, sum of Y
 0320 PRINT "DO YOU WISH TO SEE SPREAD OF VALUES?"
 0330 INPUT B$:PRINT B$
 0340 IF B$="NO" THEN S=1:GOTO 370      If only value of 'const' required, set S=1
 0350 PRINT "INPUT SCALE FACTOR (TRY 50)"
 0360 INPUT S
 0370 REM USED TO POSITION DISPLAY
 0380 REM     N  MOVES     THEORETICAL PREDICTION IS
 0390 REM      AVERAGE DIST = CONST * SQR(N)
 0400 REM    VALUE OF CONST AVERAGED OVER ALL N AND ALL K TRIALS
 0410 REM IS PRINTED (LINE 500)
 0420 REM    LINE 400 SHOWS SPREAD OF VALUES OF 'CONST'
 0430 REM    FOR EACH N AVERAGED OVER K TRIALS  (PRINTED ONLY IF REQUESTED)
 0440 FOR T=1 TO 100                    after T steps, for each value of T
 0450 Y=A(T)/K/SQR(T)*S                 scaled value of (average value of abs distance after T moves)/√T
 0460 Z=Z+Y                             cum sum of Y
 0470 IF B$="NO" THEN 490
 0480 PRINT TAB(Y);"*"                  plot * of a distance magnitude of Y - and so variability
 0490 NEXT T
 0500 PRINT S/100/S;" AVERAGE VALUE OF CONSTANT"     average of Y, with scale factor removed.
 0510 END
```

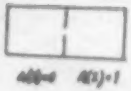
Variables

K = no. of replns. (control variable T)
P = prob. of movement L
A$ = YES for detail display
A(T) holds current sum of abs X for each T
X = distance from 0
Z = cum sum of Y
B$ = YES for display of spread
S = scale factor for display (to be removed)
Y = measure of (average abs. dist. after T moves) / √T

EHR

```
0010 REM EHRENFEST SIMULATION OF GAS DIFFUSION
0020 REM NUMERICAL OUTPUT
0030 PRINT "TYPE VALUES OF N AND K: N*K TRIALS EVERY KTH PRINTED"
0040 INPUT N,K
0050 PRINT N,K
0060 RANDOMISE
0070 S=1000
0080 REM S = NO OF PARTICLES IN LHS
0090 DIM A(1000)
0100 REM SET ALL 1000 PARTICLES IN LHS
0110 FOR I=1 TO 1000:A(I)=0:NEXT I
0120 FOR S1=1 TO N
0130 REM N*K TRIALS
0140 FOR S1 =1 TO K
0150 REM PRINT EVERY KTH VALUE OF S
0160 REM SELECT RANDOM PARTICLE
0170 J=INT(RND*1000+1)
0180 REM IF IN LHS, MOVE R AND DECREASE S
0190 IF A(J)=0 THEN S=S-1:GOTO 220
0200 S=S+1    ∴ ∉ LHS, ∴ move L
0210 REM PARTICLE MOVES
0220 A(J)=1-A(J)
0230 NEXT S1
0240 PRINT S;    ∴ ∈ LHS after every K^th move
0250 NEXT S
0260 PRINT
0270 END
```

A(I) = 0 / 1 ⇒ particle in L / R } compartment.



Variables

K interval of printing results (control variable Z1)
N N*K = no. of trials (control variable Z)
S ∴ ∈ LH compartment
A(I) location of each molecule ⟵ = 0 for LHS
= 1 for RHS
J random choice label of molecule.

Note: N*K = 2000 for "equilibrium"
Take eg. N = 125, K = 16

EHR1

graphical version of gas diffusion

```
 0010 REM EHRENFEST SIMULATION OF GAS DIFFUSION
 0020 REM GRAPHICAL VERSION
 0030 PRINT "TYPE VALUES OF N AND K: N*K TRIALS EVERY KTH PRINTED"
 0040 INPUT N,K
 0050 PRINT N,K
 0060 RANDOMISE
 0070 S=1000
 0080 REM S=NO OF PARTICLES IN LHS
 0090 PRINT "GRAPHICAL OUTPUT WANTED?:TYPE YES/NO"
 0100 INPUT A$
 0110 DIMA(1000)
 0120 FOR I=1 TO 1000:A(I)=0:NEXT I
 0130 REM SET ALL 1000 PARTICLES IN LHS
 0140 FOR E=1 TO N
 0150 REM N*K TRIALS
 0160 FOR S1 =1 TO K
 0170 REM PRINT EVERY KTH VALUE OF S
 0180 REM SELECT RANDOM PARTICLE
 0190 J=INT(RND*1000+1)
 0200 REM IF IN LHS,MOVE R AND DECREASE S
 0210 IF A(J)=0 THEN S=S-1:GOTO 240
 0220 S=S+1
 0230 REM PARTICLE MOVES
 0240 A(J)=1-A(J)
 0250 NEXT S1
 0260 IF A$="NO" THEN 340
 0270 Y=(S-450)/10
 0280 IF Y<=6 THEN 310
 0290 PRINT TAB(5);"I";TAB(Y);"*"
 0300 GOTO 350
 0310 IF Y=6 THEN PRINT TAB(5);"*":GOTO 350
 0320 PRINT TAB(Y);"*"
 0330 GOTO 350
 0340 PRINT S;
 0350 NEXT E
 0360 PRINT
 0370 END
```

Variables - see EHR

A$ yes/no for graphical output

Y scaled value of S, for TAB printing

($Y = \frac{S-450}{10}$, $Y_{max} = 55$, $Y_{min} = 5$, $Y=0$ for $S \leq 450$ (unlikely))

HERM

Epidemic - or spread of a rumour

```
0010 REM EPIDEMIC AMONG ISLAND HERMITS
0020 PRINT "INPUT THE NUMBER OF HERMITS ON THE ISLAND"
0030 INPUT K
0040 DIM A(30)
0050 FOR Z = 1 TO 10                    repetitions of expt.
0060 RANDOMIZE
0070 REM NO HERMIT INFECTED INITIALLY
0080 FOR I=1 TO K                       } set
0090 A(I)=0                             }
0100 NEXT I
0110 REM START WITH 1                   }
0120 J=1
0130 REM CHOOSE RANDOM HERMIT INFECTED  }
0140 I=INT(K*RND+1)                     <-
0150 IF I=J THEN 140                    ---- (must choose a different one)
0160 IF A(I)=1 THEN 210
0170 J=I
0180 A(I)=1    label infected hermit
0190 GOTO 140                           ->
0200 REM CHOSEN HERMIT IS IMMUNE: EPIDEMIC DIES OUT  } initialise number
0210 S=0                                             } catching disease.
0220 PRINT "OUTPUT"
0230 REM STATUS OF EACH HERMIT : 1 = INFECTED
0240 FOR I=1 TO K
0250 PRINT A(I);
0260 S=S+A(I)
0270 NEXT I
0280 PRINT
0290 PRINT S;" CAUGHT DISEASE"
0300 PRINT
0310 NEXT Z
0320 END
```

Variables :-

K number of hermits

A(I) status of hermit I , A(I) = 1 ⇒ infected
                            0 ⇒ uninfected

Z control variable for repetitions of expt.

J A(J) = hermit currently infected - active

I control variable for hermits

S no. infected (total) . Why 1st hermit does not count as infected, unless infected later.

HERM1

Epidemic - or spread of a rumour (brief version).

```
0010 REM EPIDEMIC AMONG ISLAND HERMITS
0020 REM 2ND VERSION - NO DETAIL
0030 PRINT "INPUT THE NUMBER OF HERMITS ON THE ISLAND"
0040 INPUT K
0050 PRINT K
0060 PRINT "NO OF TRIALS?"
0070 INPUT T
0080 PRINT T
0090 DIM A(30)
0100 C=0
0110 FOR S=1 TO T
0120 RANDOMISE
0130 REM ALL UNINFECTED
0140 FOR I=1 TO K
0150 A(I)=0
0160 NEXT I
0170 J=1
0180 REM CHOOSE RANDOM HERMIT FOR INFECTION
0190 I=INT(K*RND+1)
0200 IF I=J THEN 190
0210 IF A(I)=1 THEN 260
0220 J=I
0230 A(I)=1
0240 GOTO 190
0250
0260 REM HERMIT IMMUNE
0270 S=0
0280 FOR I=1 TO K
0290 S=S+A(I)
0300 NEXT I
0310 PRINT S;
0320 C=C+S
0330 NEXT S
0340 PRINT "AVERAGE NUMBER CATCHING DISEASE = ";C/T
0350 PRINT
0360 END
```

must choose different one

label infected hermit

total no. infected

cumulative no. infected.

Variables:- (in HERM1)

T = no. of trials

C = cumulative number of infected hermits.

PINBALL (Sm p26, Kg (ii)).
(Simulation of a pinball machine).

```
 0010 RANDOMISE
 0020 REM ALTER GOTOS TO 360 IN LINES 180 AND 210 TO GIVE
 0030 REM SIMPLE VERSION (IN WHICH C=B=F=G)
 0040 DIM X(20),Y(20)
 0050 REM SCORES, GAME LENGTHS
 0060 N=100
 0070 REM NO OF GAMES
 0080 REM CUMULATIVE GAME LENGTH,CUMULATIVE SCORE
 0090 I1=1
 0100 S1=0
 0110 REM INITIALISE ABSORPTIONS
 0120 E=0:F=0:G=0
 0130 FOR Z=1 TO N
 0140 REM N EXPTS
 0150 S=0:I=0
 0160 REM GAME LENGTH AND SCORE
 0170 REM A
 0180 IF RND>0.5 THEN 290
 0190 REM B
 0200 S=S+10:I=I+1
 0210 IF RND>0.5 THEN 280
 0220 REM D
 0230 S=S+10:I=I+1
 0240 GOTO 340
 0250 REM C
 0260 S=S+30:I=I+1
 0270 IF RND>1/3 THEN 310
 0280 REM E
 0290 S=S+20:I=I+1:E=E+1
 0300 GOTO 380
 0310 REM F
 0320 S=S+10:I=I+1:F=F+1
 0330 GOTO 380
 0340 REM D
 0350 IF RND>5/6 THEN I=I+1:GOTO 170
 0360 REM G
 0370 I=I+1:G=G+1
 0380 REM END OF GAME
 0390 S1=S1+S
 0400 I1=I1+I
 0410 REM SCORES AND LENGTHS
 0420 REM DISTRIBUTIONS OF EACH
 0430 X(S/10)=X(S/10)+1
 0440 Y(I)=Y(I)+1
 0450 NEXT Z
 0460 PRINT "DISTRIBUTION OF SCORES"
 0470 FOR I=1 TO 10:PRINT I;" ";:NEXT I
 0480 PRINT
 0490 MATPRINT X;
 0500 REM SCORES
 0510 PRINT
 0520 PRINT "DISTRIBUTION OF GAME LENGTHS"
 0530 MATPRINT Y;
 0540 PRINT S1/N;" =AVERAGE SCORE"
 0550 PRINT I1/N;" =AVERAGE GAME LENGTH"
 0560 PRINT "E= ";E;" F= ";F;" G= ";G
 0570 REM ABSORPTION FREQUENCIES
 0580 PRINT
 0590 END
```

after goals & pins
single session (in which C=B=F=G).

Variables:-
N = no of games, controlled by Z
I1 = cumul. game length
S1 = cumul. score
E,F,G = no. of absorptions in each state
S = score
I = current game length
X() = dist^n of score/10
Y() = dist^n of game lengths

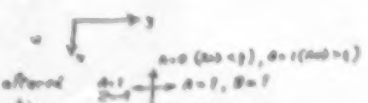
BROWNN

Simulation of Brownian movement.

```
0010 DIM A$2(36,36)
0020 RANDOMISE
0030 FOR I=0 TO 36
0040 FOR J=0 TO 36
0050 A$(I,J)=" "
0060 NEXT J
0070 A$(I,36)="*"  } to give edges
0080 A$(I,1)="*"
0090 NEXT I
0100 A$(18,18)="00"          (Origin)
0110 PRINT "COORDINATES EVERY 5 MOVES"
0120 X=20:Y=20
0130 B1=0
0140 FOR I=1 TO 19
0150 FOR K=1 TO 5
0160 B1=B1+1
0170 A =INT(2*RND)      } A=0 ⇒ 1  or  sub <, > by 1
0180 B=2*INT(2*RND)-1   } B=-1 ⇒ 1  ..........
0190 X=X+A*B (vert.)
0200 REM Y CHANGE IMPLIES  ***VERTICAL *** MOTION
0210 Y=Y+(1-A)*B (horiz).
0220 NEXT K
0230 IF (18-ABS(X-18))*(18-ABS(Y-18))<0 THEN310   off edge of display
0240 B2=INT(B1/10)
0250 B3=B1-B2*10
0260 REM CONVERT TO STRING REPRESENTATION
0270 A$(X,Y)=CHR$(B2+48)&CHR$(B3+48)
0280 PRINT "(";X;Y;")",                  print coords of every 5th point.
0290 REM  NOTE   COORDS ARE (VERT,HORIZ)
0300 NEXT I
0310 MATPRINT A$;   'Picture' of walk, every 5th point.
0320 END                  Note  earlier displayed pairs may be overwritten.
```

Variables :-

A$	display value No
B1	number of moves made.
B2	tens } digit of B1
B3	units }
X	new coord (vertical)
Y	old. coord (horizontal)
A	0 ⇒ Y altered, 1 ⇒ X altered
B	determines direction of motion



DICE6 (Gives either theoretical or experimental distribution of sum of scores on 6 dice) (1296 throws)

```
0010 FILES #1,"*"
0020 DIM N(25)
0030 FOR I =4 TO 24
0040 N(I)=0
0050 NEXT I
0060 GOTO 250
0070 REM TO PRINT EXPECTED FREQ
0075 PRINT #1,"EXPECTED FREQ","(1296 THROWS)"
0080 FOR X=1 TO 6
0090   FOR Y=1 TO 6
0100     FOR Z=1 TO 6
0110       FOR U=1 TO 6
0120         S=U+Z+Y+X
0130 N(S)=N(S)+1
0140       NEXT U
0150     NEXT Z
0160   NEXT Y
0170 NEXT X
0180 FOR I=4 TO 24
0190 PRINT #1,N(I);
0200 NEXT I
0210 PRINT #1,
0220 END
0230
0240 REM ALTERNATIVE PROGRAM (OBSERVED FREQ)
0250 PRINT #1,"INPUT NO. OF THROWS"
0260 INPUT T
0270 RANDOMISE
0280 FOR I=1 TO T
0290 S=INT(6*RND)+INT (6*RND)+INT(6*RND)
0300 S=S+INT(6*RND)+4
0310 N(S)=N(S)+1
0320 NEXT I
0330 REM END OF EXPT
0340 PRINT #1,"OBSERVED FREQS (";T;"THROWS)"
0350 GOTO 180        (for output)
```

To go direct to expt: (Runs 1.66 if expected frequencies are required)

sum of scores frequency chart

output frequencies

sum of scores frequency chart

output for expt.

Variables

N. array of frequencies
I control variable for N
X, Y, Z, U scores on 4 dice
S sum of scores
T no. of throws

NCOINS

Simulates tossing of N coins simultaneously, T times

```
0010 FILES #1,"*"
0020 PRINT #1," INPUT NO. OF COINS AND NO. OF THROWS"
0030 INPUT N,T
0040
0050 FOR I =1 TO N+1
0060 A(I)=0
0070 NEXT I
0080 RANDOMISE
0090
0100 FOR J =1 TO T
0110 S=0
0120 FOR I=1 TO N
0130 X =INT (0.5+RND)      0,1 (1 = 'head')
0140 S=S+X                 number of 'heads'
0150 NEXT I
0160
0170 A(S+1)=A(S+1)+1       frequency classification
0180 NEXT J
0190
0200 PRINT #1,"RESULTS OF",T,"THROWS OF",N,"COINS"
0210 PRINT #1,"NO. OF HEADS",
0220 FOR I=1 TO N+1        } print heading
0230 PRINT #1,I-1,         }
0240 NEXT I
0250 PRINT #1,
0260
0270 PRINT #1,"            ";    (spaces for alignment)
0280 FOR I=1 TO N+1        } output frequencies
0290 PRINT #1,A(I),        }
0300 NEXT I
0310 PRINT #1,
0320 END
```

Variables

T = no. of throws
N = no of coins
A() frequency classes
J, I control variables for T, N
X result of single toss
S = no. of heads

output

DECAY2

```
0010 REM 2 STAGE DECAY U-->V-->W
0020 PRINT "INPUT NUMBER OF SECONDS TO RUN"
0030 INPUT N:PRINT N
0040 PRINT "INPUT NUMBER OF SUBDIVISIONS PER SECOND"
0050 INPUT T:PRINT T
0060 PRINT "INPUT REACTION COEFFICIENTS K1,K2,ON SAME LINE WITH COMMA BETWEEN"
0070 REM DU/DT =-K1*U
0080 REM DV/DT =-K2*V+....
0090 INPUT K1,K2:PRINT K1,K2
0100 PRINT "AMOUNTS AT END OF EACH SECOND"
0110 REM INITIALISE
0120 U=1
0130 V=0
0140 W=0
0150 M=0
0160 REM MAX V=M
0170 PRINT U,V,W
0180 D=1/T
0190 FOR J=1 TO N
0200 FOR I=1 TO T
0210 REM ONE ITERATION T TIMES/SEC
0220 U1=U-K1*U*D           }
0230 V1=V+K1*U*D-K2*V*D    } updated values of U,V,W
0240 W1=W+K2*V*D           }
0250 IF V1>M THEN M=V1     update M
0260 U=U1:V=V1:W=W1        update
0270 NEXT I
0280 PRINT U,V,W
0290 NEXT J
0300 PRINT "MAX V WAS ";M
0310 PRINT
0320 END
```

Variables

U, V, W amounts of substances (initially 1, 0, 0)
N = no. of 'seconds' = no. of results printed out
T = no. of steps per 'second'; D = 1/T
K1, K2 reaction coeffs
$\frac{dU}{dt} = -K1 \cdot U$, $\left(\frac{dV}{dt}\right) = -K2 \cdot V$ (usual)
M = max. value of V
J = control variable for N
I = " " " " T
U1, V1, W1, updated values of U, V, W.

RUMOUR (Simulates spread of a rumour in a population of (N+1) people)

```
0010 FILES #1,"**"
0020 PRINT #1,"INPUT N(PEOPLE)";
0030 PRINT #1,"AND P (RATE OF SPREAD)"
0040 INPUT N,P
0050 P=P/N                P is scaled, so that equations of lines 160,150 are dimensionless
0060 PRINT #1, "N=";N; " P=";P                (we are effectively dealing with proportions).
0070 PRINT #1,"FRACTION OF POPULATION NOT HEARING RUMOUR  IS CALCULATED"
0080 PRINT #1,"(FOR 4 SUCCESSIVE CALCULATIONS, STEP WIDTHS HALVED EACH TIME)"
0090 REM GIVES SUCCESSIVELY IMPROVED ACCURACY AS STEP LENGTHS DECREASED
0100 FOR K=1 TO 4
0110 PRINT #1, "    X","Y"          heading.
0120 X=N
0130 Y=1
0140 X1=X-P*X*Y
0150 Y=Y-P*Y*(N-2*X)
0160 IF X>2 THEN 180          print only 1st 2 pairs of X,Y pairs. († see below)
0170 PRINT USING "#####.##",#1,X1,Y          round X, Y
0180 IF (Y-N)*X<0 THEN 210          if not, Y>N or X<0 - and stop.
0190 PRINT #1,"P IS TOO LARGE FOR ACCURACY OF THE STEP BY STEP METHOD"
0200 GOTO 330 ——▷ (1)
0210 REM UPDATE X
0220 X=X1
0230 REM TEST FOR ZERO INFECTIVES
0240 IF Y>=1 THEN 40 ——▷ (2)
0250 IF Y<0 THEN PRINT #1, "RESULT IS INACCURATE: STEP LENGTH TOO BIG":GOTO 290
0260 REM CASE 0<Y<1
0270 PRINT
0280 PRINT #1,X/N;" -FRACTION NOT HEARING"
0290 PRINT#1,
0300 REM HALVE STEP WIDTH
0310 P=P/2
0320 NEXT K
0330 END
0340
```

(2) 3)

Variables
N+1 = no of people
P coeff. in $\frac{dx}{dt} = -kx(N-x)$ and $\frac{dy}{dt} = kxy$
K = control variable for step accuracy
X, Y, = uninfected, infecting

(†) If P is small, perhaps delete 0170 (since not put will be length)
If P is too large, accuracy will be poor – see P. 180, 190

FOX The fox starts at (X,Y) and chases a hare, who runs up the Y axis from (0,0)

```
0010 FILES #1,"**"
0020 PRINT #1,"INPUT FOX COORDS,SPEEDS F/H,MAX DIST"
0030 REM STOP IF DIST >M OR <1
0040 INPUT X,Y,Q,P,M
0050 REM FOX STARTS AT X,Y;  HARE STARTS AT (0,0)
0060 Z=0          initial position of hare
0070 D=SQR(X*X+(Y-Z)*(Y-Z))          positions and distance.
0080 PRINT #1,X,Y,Z,D
0090 X=X-Q*X/D  } fox
0100 Y=Y+Q*(Z-Y)/D } hare } - new positions
0110 Z=Z+P
0120 IF (D-1)*(D-M)>0 THEN 140          - if the chase is on for
0130 GOTO 70
0140 END
```

Variables
X, Y initial coords of fox
Q, P speeds of fox and hare
M max permitted distance between f. & h.
Z y-coordinate of hare
D distance between f. & h.

FOXRAB (Simulation of interaction between fox/rabbit populations).

```
0010 REM FOXRAB
0020 REM DX/DT=A*X*(1-Y)   ) SCALED UNITS
0030 REM DY/DT=C*Y*(X-1)   )  FOR X,Y
0040 READ A,C
0050 DATA 2,1
0060 REM TIME STEP = H: EVERY MTH POINT PLOTTED
0070 REM T=MAX NUMBER OF POINTS PLOTTED: EQUILIBRIUM POSITION (1,1)
0080 READ H,M,T
0090 DATA 0.01,10,60
0100 REM BEGIN CURVE-PLOT
0110 READ X0,Y0
0120 IF X0<0 THEN 350
0130 FOR Z=1 TO T
0140 PRINT X0;Y0
0150 FOR I=1 TO M
0160 REM GRADIENTS AT POINT
0170 D1=A*X0*(1-Y0)
0180 D2=C*Y0*(X0-1)
0190 REM PREDICT NEW X,Y
0200 X1=X0+D1*H
0210 Y1=Y0+D2*H
0220 REM GRADIENT AT NEW X,Y
0230 D3=A*X1*(1-Y1)
0240 D4=C*Y1*(X1-1)
0250 REM CORRECTED NEW X,Y
0260 X0=X0+(D1+D3)*H/2
0270 Y0=Y0+(D2+D4)*H/2
0280 NEXT I
0290 NEXT Z
0300 PRINT
0310 PRINT
0320 PRINT
0325 GOTO 110
0330 DATA 0.25,0.25,0.5,0.5,0.75,0.75
0340 DATA -1,-1
0350 END
```

(2) → 110 (IF X0<0 THEN 350 ——→ (1) to signal end of data.)

140 current position, printed every Mth point

GRADIENTS AT POINT } $\frac{d}{dt}$ at (X_0, Y_0)

PREDICT NEW X,Y } predicted value after time interval h

GRADIENT AT NEW X,Y } $\frac{d}{dt}$ at predicted point

CORRECTED NEW X,Y (TRAPEZIUM METHOD) } corrected 'new' point using average of gradients

325 GOTO 110 ——→ (2)

(1) → 350 END

Variables

A,C constants in differential equations

H - time step, N, every Mth

T - max no. of points plotted

(X_0, Y_0) current position (relative, from 1,1)

Z, I control variables for T, M

D1, D2 : $\frac{dx}{dt}$, $\frac{dy}{dt}$ at (X_0, Y_0)

$(X1, Y1)$ predicted point

D3, D4 : $\frac{dx}{dt}$, $\frac{dy}{dt}$ at $(X1, Y1)$

FOXEC

Simulation of predator/prey [fox/rabbits] - graphical output

```
0010 DIM A$(60,60)
0020 REM FOXRAB
0030 REM DX/DT=A*X*(1-Y)    ) SCALED UNITS
0040 REM DY/DT=C*Y*(X-1)    ) FOR X,Y
0050 READ A,C
0060 DATA 2,1
0070 REM TIME STEP = H: EVERY NTH POINT PLOTTED
0080 REM T=MAX NUMBER OF POINTS PLOTTED: EQUILIBRIUM POSITION (1,1)
0090 READ H,N,T
0100 DATA 0.01,10,40
0110 REM BEGIN CURVE-PLOT
0120 READ X0,Y0
0130 FOR I=0 TO 60
0140 FOR J=0 TO 60
0150 A$(I,J)=" "
0160 NEXT J,I
0170 IF X0<0 THEN 670
0180 READ S1,S2
0190 REM S1,S2 ARE SCALE FACTORS
0200 REM FOR X(VERT) AND Y(HORIZ) AXES
0210 FOR Z=1 TO T
0220 X2=INT(X0*S1):Y2=INT(Y0*S2)
0230 Z2=Z
0240 IF Z2<10 THEN 270
0250 Z2=Z2-10
0260 GOTO 240
0270 A$(X2,Y2)=CHR$(Z2+48)
0280 REM LOOP TO CALCULATE N NEW POINTS
0290 FOR I=1 TO N
0300 REM GRADIENTS AT POINT
0310 D1=A*X0*(1-Y0)
0320 D2=C*Y0*(X0-1)
0330 REM PREDICT NEW X,Y
0340 X1=X0+D1*H
0350 Y1=Y0+D2*H
0360 REM GRADIENT AT NEW X,Y
0370 D3=A*X1*(1-Y1)
0380 D4=C*Y1*(X1-1)
0390 REM CORRECTED NEW X,Y (TRAPEZIUM METHOD)
0400 X0=X0+(D1+D3)*H/2
0410 Y0=Y0+(D2+D4)*H/2
0420 X2=INT(X0*S1):Y2=INT(Y0*S2)
0430 IF A$(X2,Y2)=" " THEN A$(X2,Y2)="."
0440 NEXT I
0450 NEXT Z
0460 REM BEGIN DISPLAY
0470 FOR J=1 TO 60 :PRINT " ";:NEXT J
0480 FOR I=1 TO 5:IF I*81>59 THEN 490:A$(I*81,0)=CHR$(I+48):NEXT I
0490 FOR J=1 TO 3:IF J*82>60 THEN 500:A$(0,82*J)=CHR$(J+48):NEXT J
0500 PRINT
0510 FOR I=0 TO 60
0520 FOR J=0 TO 60
0530 PRINT A$(I,J);
0540 NEXT J
0550 PRINT
0560 NEXT I
0570 PRINT
0580 PRINT
0590 PRINT
0600 GOTO 120
0610 DATA 0.25,0.25,10,15
0620 DATA 0.5,0.5,20,30
0630 DATA 0.75,0.75,30,40
0640 DATA .8,.8,40,40
0650 DATA .35,.35,19,20
0660 DATA -1,-1
0670 END
```

Variables

A,C constants in diff. equations

H = time step, N every Nth

T = max no. of points plotted

(X,Y) current state (rabbits, foxes)

Z, Z2 control variables for T, N.

S1, S2 : $\frac{X}{X_0}$, $\frac{Y}{Y_0}$ of (X_0, Y_0)

(X1, Y1) predicted point

S3, S4 : $\frac{X}{X_1}$, $\frac{Y}{Y_1}$ of (X_1, Y_1).

① Read scale factors for next curve

② Repeat calculation for another starting point

0.25, 0.25 will scale factors 10, 15 for X, Y axes respectively

PROJ.

```
0010 REM PROJ WITH AIR RESISTANCE
0020 PRINT "HORIZ,VERT  SPEEDS"
0030 INPUT U0,V0
0040 PRINT U0,V0
0050 PRINT " NO. OF SUBDIVISIONS PER SEC. "
0060 INPUT D
0070 PRINT D
0080 PRINT "NO. OF SECONDS TO RUN"
0090 INPUT T
0100 PRINT T
0110 PRINT "OUTPUT EVERY N SECONDS -- INPUT N "
0120 INPUT N
0130 PRINT N
0140 REM ACCN PROPL V*V, CONST K
0150 REM ACCN DUE TO GRAVITY, G
0160 K=0.003:G=9.8
0170 REM INITIALISE
0180 X0=0
0190 Y0=0
0200 N1=0
0210 N2=INT(N*D+0.0001)
0220 REM TO ALLOW N TO BE NON-INTEGER OR<1
0230 PRINT "X VALUE        Y VALUE          TIME (SEC)"
0240 DIM P1$80
0250 P1$="###.##,5S,###.##,10S,##.##"
0260 FOR I=1 TO N2
0270 W0=SQR(U0*U0+V0*V0)
0280 REM SPEED AT 0
0290 E0=-K*W0*U0
0300 REM ACCNS AT 0
0310 F0=-G-K*W0*V0
0320 U1=U0+E0/D
0330 REM 1ST EST SPEEDS AT 1
0340 V1=V0+F0/D
0350 REM 1ST EST (VEL) AT 1
0360 W1=SQR(U1*U1+V1*V1)
0370 E1=-K*W1*U1
0380 REM ACCNS AT 1
0390 F1=-G-K*W1*V1
0400 U1=U0+(E1+E0)/2/D
0410 REM CORRECTED SPEEDS AT 1
0420 V1=V0+(F1+F0)/2/D
0430 X0=X0+(U1+U0)/2/D
0440 REM CORRECTED POS AT 1
0450 Y0=Y0+(V1+V0)/2/D
0460 U0=U1
0470 REM UPDATE
0480 V0=V1
0490 IF Y0=<0 THEN 540
0500 NEXT I
0510 PRINT USING P1$,X0,Y0,N1+N
0520 IF N1<T THEN N1=N+N1:GOTO 260
0530 GOTO 550
0540 PRINT "X=";X0;" Y= ";Y0;" AFTER ";N1+I/D;" SEC"
0550 PRINT
0560 END
```

for layout of output
eg 6 of N sec, 5 calc" per sec.

Y is now -ve

Time if flight not terminated.

Variables

U0, V0 initial speeds
X0, Y0 position
D, T subdivisions per sec, seconds to run.
N print interval.
K set at 0.003 ; acc" $-kV^2$
G 9.8 acc" due to gravity.
N1 time since start of beginning of current loop
N2 no of subdivisions per loop (= N*D)
outer variable I
W0 initial (velocity)
E0, F0 initial acc"
U1, V1, W1, E1, F1, values at end of interval
P1$ "picture" for output (PRINT USING)

BIRTH

```
0005 RANDOMISE
0010 REM BDATE
0020 DIM A(366),B(101),C(101)
0030 MAT B=ZER(101)
0040 FOR E=1 TO 100
0050 MAT A=ZER(366)
0060 FOR I= 1 TO 100
0070 REM SELECT NEW BIRTH DATE
0080 X= INT(RND*365.25)+1
0090 REM A(X)=1 => BIRTHDATE ALREADY SELECTED
0100 IF A(X)=1 THEN 150
0110 A(X)=1
0120 NEXT I
0130 REM NO COINCIDENCE YET : SET I ="101"
0140 B(101)=B(101)+1:GOTO 170
0150 B(I)=B(I)+1
0160 REM FREQ DISTN. OF GROUP SIZE
0170 NEXT E
0180 PRINT "VALUE OF E                EXPTL PROBY.
0190 C(0)=0
0200 D=1
0210 FOR N=0 TO 70
0220 C(N+1)=C(N)+B(N+1)
0230 REM CUMULATIVE FREQ
0240 U=D*(365-N)/365
0250 REM THEORETICAL PROBY.  "NO"  COINCIDENCE
0260 REM PRINT EVERY 5TH RESULT
0270 IF N<>5*INT(N/5) THEN 330
0280 PRINT N+1,
0290 PRINT "       ",
0300 PRINT USING "10.2#",C(N+1)/100,
0310 PRINT "       ",
0320 PRINT USING "10.3#",1-U
0330 NEXT N
0340 END
```

Variables

A() no. of days of year
B() freq. distr. of group size needed for coincidence
C() cum. distr. of B()
Z control variable, no. of expts
N control variable, no. in group
U theoretical prob. of no coincidence.
I control variable. (Indicates group needed for I = 1 & 1 pp. with a date coincidence; on exit I = pp. size to produce first coincidence – is set at default value 101)

output in column layout

THEOR.. PROBY."

NEUTRON

```
0010 REM NEUTRON ABSORPTION
0020 RANDOMISE
0030 S=2
0040 REM THICKNESS OF SHIELDING
0050 L=5
0060 REM "LIFE" OF NEUTRON BEFORE ABSORPTION/ESCAPE
0070 K=0        no. of escapes
0080 FOR I=1 TO 100
0090 REM 100 EXPTS
0100 X=0:Y=0
0110 FOR K=1 TO L          L = "life time"
0120 IF RND>0.8 THEN 180
0130 REM CORRESPONDS TO "STAY PUT"
0140 A=INT(2*RND)                    A=0,1
0150 B=2*INT(2*RND)-1      Move      B=-1,1
0160 X=X+A*B
0170 Y=Y+(1-A)*B
0180 NEXT K
0190 IF ABS(X)>S-1 THEN E=E+1:GOTO 210
0200 IF ABS(Y)>S-1 THEN E=E+1
0210 NEXT I
0220 PRINT "PROBABILITY OF ESCAPE =  ";E/100
0230 PRINT "S= ";S,"L= ";L
0240 PRINT
```

S = 2

A=0,B=1 ; A=1,B=0 ; A=1,B=1 ; A=0,B=-1

Variables

S = no. of steps (in X or Y direction) to escape
L = 'life' of neutron – control variable K
E = cumulative number of escapes
I = control variable for expts.
X, Y current position of neutron
A, B determine direction of motion.

$p=\frac{1}{4}$; $p=\frac{1}{4}$; $p=\frac{1}{4}$; $p=\frac{1}{4}$

'no change', $p=\frac{1}{2}$

RDUP

```
0010 REM ROADUP - ONE WAY TRAFFIC, CONTROLLED BY LIGHTS
0020 PRINT "INPUT P1":INPUT P1:PRINT P1;
0030 PRINT
0040 PRINT "INPUT P2":INPUT P2:PRINT P2;
0050 PRINT
0060 READ T1,T2,T3,T4
0070 IF T1=0 THEN 480
0080 RANDOMISE
0090 V1=0:V2=0
0100 Q1=0:Q2=0
0110 M1=0:M2=0
0120 R1=0:R2=0
0130 REM PHASES II,III,IV END AT S2,S3,S4
0140 S2=T1+T2
0150 S3=T1+T2+T3
0160 S4=T1+T2+T3+T4
0170 FOR Z=1 TO 100
0180 FOR I = 1 TO S4
0190 REM VEH ARRIVALS
0200 IF RND<P1 THEN V1=V1+1:Q1=Q1+1
0210 IF RND<P2 THEN V2=V2+1:Q2=Q2+1
0220 IF RND<P2 THEN V2=V2+1:Q2=Q2+1
0230 IF I>T1 THEN 280
0240 REM REDUCE Q1 IN PHASE I
0250 IF Q1>1 THEN Q1=Q1-2:GOTO 280
0260 IF Q1=1 THEN Q1=0
0270 REM AMEND MAX & CUM-QUEUE AT EACH PASS
0280 IF Q1>M1 THEN M1=Q1
0290 R1=R1+Q1
0300 REM JUMP IF PHASE II OR IV
0310 IF I<S2+1 THEN 370
0320 IF I>S3 THEN 370
0330 REM REDUCE Q3 IN PHASE III
0340 IF Q2>1 THEN Q2=Q2-2:GOTO 370
0350 IF Q2=1 THEN Q2=0
0360 REM AMEND MAX & CUM-QUEUE AT EACH PASS
0370 IF Q2>M2 THEN M2=Q2
0380 R2=R2+Q2
0390 NEXT I
0400 NEXT Z
0410 PRINT
0420 PRINT  "PATTERN ";T1;T2;T3;T4
0430 PRINT M1;M2; " MAX QUEUES"
0440 PRINT V1;V2;" TOTAL VEHICLES : 100 TRIALS"
0450 PRINT R1/S4/100;R2/S4/100;" AVERAGE QUEUE LENGTHS"
0460 REM FRESH DATA?
0470 GOTO 60
0480 END
0490 DATA 1,1,1,1
0500 DATA 2,1,2,1
0510 DATA 5,1,5,1
0520 DATA 3,1,6,1
0530 DATA 0,0,0,0
```

Variables

P1, P2 prob. veh arrival S'bd/N'bd.
V1, V2 no. of veh. arrivals S'bd/N'bd.
Q1, Q2 queue lengths
M1, M2 max queue
R1, R2 cum. queue
T1, T2, T3, T4 time of phases I-IV in sec.
S2, S3, S4 time to end of phase II-IV.
Z counter for 100 cycles.
I control variable for seconds within cycle of traffic lights

RDJN

```
0010 REM ROADJUNCTION
0020 PRINT " IF DETAIL NOT REQUIRED, INPUT  N"
0030 INPUT Q$
0040 PRINT Q$
0050 RANDOMISE
0060 B9=0:A9=0
0070 A1=0:A2=0
0080 B1=0:B2=0
0090 M=0
0100 B3=0
0110 PRINT "INPUT TRAFFIC DENSITIES  A,B"
0120 INPUT A,B
0130 PRINT A,B
0140 IF Q$<>"N" THEN PRINT TAB(10);"A1";TAB(20);"A2";TAB(30);"QUEUE AT B";
0150 IF Q$<>"N" THEN PRINT TAB(60);"B ARRIVALS"
0160 FOR Z =1 TO 100
0170 A2=A1
0180 A1=0
0190 IF RND<A THEN A1=1
0200 B1=0
0210 IF RND<B THEN B1 =1
0220 B2=B2+B1
0230 IF A2+A1>0 THEN 250
0240 IF B2>0 THEN B2=B2-1:B9=B9+1
0250 B3=B3+B2
0260 IF B2>M THEN M=B2
0270 A9=A9+A2
0280 IF Q$="N" THEN 390
0290 B1$="*"
0300 IF B1=0 THEN B1$=" "
0310 A1$="*":A2$="*"
0320 IF A2=0 THEN A2$=" "
0330 IF A1=0 THEN A1$=" "
0340 Q=B2
0350 IF Q>35 THEN Q=35
0360 PRINT TAB(10);A1$;TAB(20);A2$;TAB(30); [FOR U=1 TO Q:PRINT"*";:NEXT U]
0370 IF B2>35 THEN PRINT "+";
0380 PRINT TAB(70);B1$
0390 NEXT Z
0400 PRINT A9;B9;" VEHICLES PASSING IN DIRECTIONS A & B"
0410 PRINT B3/100 ;"  = AVERAGE QUEUE"
0420 PRINT M;" =MAX QUEUE AT B"
0430 END
```

initialise

A traffic volumes

A arrival ?

B arrival ?

B queue augmented

B cannot exit

B can exit, goes to B9, queue reduced.

Cumulative exit

max

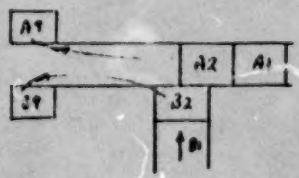
A goes to A9

(skip output)

temporary marker

& printed display modifiers

makes for overflow.



Variables

Q1$ user's response for detailed output
B9, A9 total vehs. passing thro'.
A1, A2, B1 0/1 for veh. in road section/arrivals
B2, B3, M queue, cumulative, max. queue of B
A, B probs of arrival at A, B (0≤A, B≤1)
Z control variable, 100 cycles.
B1$, A1$, A2$ "pictures" of B1, A1, A2.
Q temporary marker for B2, restricted ≤ 0035
(only necessary if A, B are large).

HISTOG

```
0001 DIM H(100),K(100)
0002 REM HISTOGRAM FRAMEWORK:LINES 2-9 AND 1000-> RESERVED
0003 PRINT "INPUT WIDTH OF RESULTS CLASSES"
0004 REM A1*B1 MUST NOT EXCEED 100. TYPICALLY 20*5
0005 INPUT B1:PRINT B1
0006 PRINT "INPUT NO OF RESULTS CLASSES"
0007 INPUT A1
0008 PRINT A1
0009 N=A1*B1:MAT H=ZER(N):MAT K=ZER(A1):V=0
0020
1000 REM HISTOGRAM SUBROUTINE
1010 REM FREQUENCIES HELD IN H(N), GROUPED IN K(A1)
1020 PRINT "HISTOGRAM OF RESULTS"
1030 PRINT
1040 Y1=0:Y2=0
1050 Y0=0
1060 FOR J=1 TO A1
1070 K(J)=0
1080 FOR X=1 TO B1
1090 X=B1*(J-1)+X
1100 K(J)=K(J)+H(X)
1110 Y0=Y0+H(X)
1120 Y1=Y1+H(X)*X
1130 Y2=Y2+X*X*H(X)
1140 NEXT X
1150 PRINT (J-1)*B1+1;"TO";J*B1;TAB(12);
1160 B2=K(J)
1170 FOR L=1 TO B2:PRINT "*";:NEXT L
1180 PRINT
1190 NEXT J
1200 PRINT
1210 IF V=1 THEN PRINT "** NB OVERFLOW HAS OCCURRED **":PRINT
1220 PRINT "MEAN = ";Y1/Y0
1230 PRINT "S.D. = ";SQR(Y2/Y0-Y1*Y1/Y0/Y0)
1240 RETURN
```

Initialise H, K
Set overflow indicator to zero.
Heading
Initialise Σf, Σxf, Σx²f
each grouped class
initialise
index X
add freq. dist'n class
Σf
Σxf
Σx²f
Legend.
'bar'

Variables

- H( ) freq classes built up for original observations
- K( ) results classes - grouped frequencies
- A1 no. of freq. classes (grouped) - control variable J
- B1 width ——— K
- N = A1 × B1 = no. of (original) freq. classes (≤100 here)
- V overflow indicator. (Originally set to 0; set to 1 in second program if X value too large for keeping array, i.e. > 100)
- Y0 = Σf
- Y1 = Σxf
- Y2 = Σx²f
- B2 = current class frequency - control variable L.

$\frac{\Sigma xf}{\Sigma f}$

$\sqrt{\frac{\Sigma x^2 f}{\Sigma f} - \text{C.F.}^2}$

This program is a 'framework', which may be added to other programs, such as COLLGEN, to produce a histogram of results in grouped frequency classes. (In some systems a command MERGE may be used to amalgamate two BASIC programs). In each case a few additional instructions must be added (for an example, see below).

Method of operation

A frequency classification is copied into the array H, of dimension N. Classes are then grouped together in a new array K, to give A1 classes of width B1 (where N = A1 x B1). A histogram is output, together with the mean and standard deviation of the frequency classification. H and K are initially given the dimension 100 but are redimensioned to N and A1, (A1 and B1 are input).

Details of derivation of COLLGEN1 from COLLGEN and HISTOG

Note that the line numbers have been rationalised by the use of a RESEQUENCE command.

Line 10 of COLLGEN is replaced by line 90 of COLLGEN1.

Line 40 is replaced by new lines 120, 130, giving V the dimensions of N.

Lines 380-400 are deleted, and lines 470-480 of COLLGEN1 are added, to call the histogram subroutine. The overflow check, setting V = 1, is already present in line 300 of COLLGEN, but otherwise would need to be added (to prevent array overflow, and/or faulty printing of the histogram). BEETLE, FIVE8 and RUN81 were produced in this way (but note that RUN81 has been considerably simplified and reorganised).

FIG 6 H

Collecting P 'S's and Q 'F's : graphical output.

```
0010 DIM N(100),K(20)
0020 REM USES HISTOGRAM FRAMEWORK
0030 A1=20          no. of groups (freq. classes ('rows')
0040 B1=5           width of freq. classes
0050 N=A1*B1:MAT N=ZER(N):MAT=ZER(A1)
0060 RANDOMISE
0070 DIM F(100)                    used to collect freq. of waiting times
0080 MAT F=N
0090 PRINT "INPUT P,Q"             P 'S's
0100 INPUT P,Q                     and Q 'F's
0110 PRINT P;Q
0120 K=6                           K = "goal" of each subtrial . eg K=6, p = ½
0130 PRINT "HOW MANY REPETITIONS?"
0140 INPUT M:PRINT M
0150 J=0                           initialise cumulative unit      J = ΣI
0160 PRINT "WAITTIMES"             } Heading
0170 PRINT
0180 FOR I=1 TO M                  I control variable for output.
0190 I=0            unit           I = count of trials = "waiting time"
0200 A=0            no. of 'S's    I = waiting time.
0210 B=0            no. of 'F's    A, B current no. of 'S's, 'F's
0220 I=I+1
0230 R=RND
0240 IF K*R<2 THEN 260             'S' or 'F'
0250 GOTO 220                      added
0260 IF K*R<1 THEN A=A+1:GOTO 280  'S' } formed
0270 B=B+1                         'F'
0280 IF A<P THEN 220               } if either target not
0290 IF B<Q THEN 220               attained.
0300 PRINT I;                      wait time for(P,Q). → output.
0310 IF I>100 THEN I=100
0320 F(I)=F(I)+1
0330 J=J+I                         cumulative unit
0340 NEXT I
0350 PRINT
0360 PRINT "FREQUENCIES"           }
0370 MATPRINT F;                   frequencies of waittimes
0380 MAT N=F
0390 GOSUB 430
0400 END
0410
0420
0430 REM HISTOGRAM SUBROUTINE
0440 REM FREQUENCIES HELD IN N(100), GROUPED IN K(100)
0450 PRINT "HISTOGRAM OF RESULTS"
0460 PRINT                         Y0 }
0470 Y1=0:Y2=0                     Y1 } accumulators of Σf → mean
0480 Y0=0                          Y2 } Σf² → s.d.
0490 FOR J=1 TO A1
0500 K(J)=0                        set each (grouped) freq. class to zero.
0510 FOR X=1 TO B1
0520 X=B1*(J-1)+X                  } select next X
0530 K(J)=K(J)+N(X)                } increment freq. class no.
0540 Y0=Y0+N(X)                    }
0550 Y1=Y1+N(X)*X                  } increment Σf, Σxf
0560 Y2=Y2+X*X*N(X)                }
0570 NEXT X
0580 PRINT (J-1)*B1+1;"TO";J*B1;TAB(12);   print class boundaries
0590 S2=K(J)                       }
0600 FOR L=1 TO S2:PRINT "*";:NEXT L   } output bar of histogram.
0610 PRINT
0620 NEXT J
0630 PRINT "MEAN = ";Y1/Y0                          Σxf/Σf
0640 PRINT "S.D. = ";SQR(Y2/Y0-Y1*Y1/Y0/Y0)         √(Σx²f/Σf − (Σxf/Σf)²)
0650 RETURN
```

COLLEENH

```
0010 DIM H(100),K(100)
0020 REM HISTOGRAM FRAMEWORK:
0030 PRINT "INPUT WIDTH OF RESULTS CLASSES"
0040 REM A1*B1 MUST NOT EXCEED 100. TYPICALLY 20*5
0050 INPUT B1:PRINT B1
0060 PRINT "INPUT NO OF RESULTS CLASSES"
0070 INPUT A1
0080 PRINT A1
0090 N=A1*B1:MAT H=ZER(N):MAT K=ZER(A1):V=0
0100 REM TARGET VECTOR (P,Q,R,S,...) MEANS P OF "A",Q OF "B" ETC.
0110 RANDOMISE
0120 DIM F(100)
0130 MAT F=H
0140 PRINT "INPUT DIMENSIONS OF TARGET VECTOR"
0150 INPUT K
0160 PRINT K
0170 PRINT "HOW MANY REPETITIONS?"
0180 INPUT N
0190 PRINT N
0200 I1=0
0210 PRINT "INPUT TARGET VECTOR"
0220 MATINPUT A(K)
0230 MATPRINT A ;
0240 DIM B(20)
0250 MAT B=A
0260 PRINT "WAITTIMES"
0270 PRINT
0280 FOR Z=1 TO N
0290 I=0
0300
0310
0320 I=I+1
0330
0340 R=INT (K*RND+1)
0350
0360 A(R)=A(R)-1
0370 FOR L=1 TO K:IF A(L)>0 THEN 320:NEXT L
0380 PRINT I;
0390 IF I>100 THEN I=100:V=1
0400 F(I)=F(I)+1
0410 I1=I1+I
0420 MAT A=B
0430 NEXT Z
0440 PRINT
0450 PRINT "FREQUENCIES"
0460 MATPRINT F ;
0470 MAT H=F
0480 GOSUB 520
0490 END
0500
0510
0520 REM HISTOGRAM SUBROUTINE
0530REM FREQUENCIES HELD IN H(N), GROUPED IN K(A1)
0540 PRINT "HISTOGRAM OF RESULTS"
0550 PRINT
0560 Y1=0:Y2=0
0570 Y0=0
0580 FOR J=1 TO A1
0590 K(J)=0
0600 FOR X=1 TO B1
0610 X=J*(J-1)+X
0620 K(J)=K(J)+H(X)
0630 Y0=Y0+H(X)
0640 Y1=Y1+H(X)*X
0650 Y2=Y2+X*X*H(X)
0660 NEXT X
0670 PRINT (J-1)*B1+1;"TO";J*B1;TAB(12);
0680 B2=K(J)
0690 FOR L=1 TO B2:PRINT "*";:NEXT L
0700 PRINT
0710 NEXT J
0720 PRINT
0730 IF V=1 THEN PRINT "** NB OVERFLOW HAS OCCURRED **":PRINT
0740 PRINT "MEAN = ";Y1/Y0
0750 PRINT "S.D. = ";SQR(Y2/Y0-Y1*Y1/Y0/Y0)
0760 RETURN
```

BEETLH

(Simulation of the game, with output histogram) (For matching BASIC see BEETLE, p.79)

```
0010 DIM N(100),K(100)
0020 REM HISTOGRAM FRAMEWORK:
0030 PRINT "INPUT WIDTH OF RESULTS CLASSES (<6)"
0040 INPUT B1
0050 PRINT B1
0060 PRINT "INPUT NO OF RESULTS CLASSES (<21)"
0070 INPUT A1
0080 PRINT A1
0090 N=A1*B1:MAT N=ZER(N):MAT K=ZER(A1)
0100 REM TARGET VECTOR (P,Q,R,S...) MEANS P OF "A",Q OF "B", ETC
0110 DIM F(100)      # collect frequency of (2nd half) game lengths
0120 MAT F=N         # used as histogram
0130 K=6
0140 REM 6 SIDED DIE
0150 PRINT "HOW MANY REPETITIONS?"
0160 REM NO OF GAMES
0170 REM IF N>10 ONLY GAME LENGTHS AND HISTOGRAM  FOR 2ND STAGE GIVEN
0180 INPUT N                                          (see p. 100)
0190 PRINT N
0200 IF N>10 THEN A$="NO":GOTO 240
0210 PRINT "DETAIL OF MOVES REQUIRED AT EACH STAGE?:YES/NO"
0220 INPUT A$:PRINT A$
0230
0240 I2=0    # sum of total game lengths
0250 FOR Z= 1 TO N
0260 RESTORE          # for new game
0270 I1=0             # length of (total) game
0280 B$="TO SCORE SIX"
0290 GOSUB 510 ------------> # get 1
0300 B$="TO COMPLETE GAME"
0310 GOSUB 510 ------------> # get 2, (2',1'), 4, 3, 5, 6
0320 F(I)=F(I)+1      # freq. classification for second stage waiting times
0330 IF N<11 THEN PRINT "BEETLE IN ";I1;" MOVES":PRINT   } total game length
0340 IF N>10 THEN PRINT I1;
0350 I2=I2+I1         # sum of total game lengths
0360 NEXT Z
0370
0380 PRINT
0390 PRINT "AVERAGE (TOTAL) GAME LENGTH WAS ";I2/N
0400 PRINT "RESULTS FOR 2ND HALVES OF GAMES (IE. AFTER '6')"
0410 M=0
0420 FOR I=1 TO 100:IF F(I)>0 THEN M=I:NEXT I                    } output frequency
0430 FOR I=1 TO M:PRINT F(I);:NEXT I                               distribution of
0440 REM M IS INDEX OF LARGEST NON-ZERO F: TO AVOID PRINTING EMPTY CELLS  } second halves
0450 MAT N=F          # copy F
0460 GOSUB 670 ------------> histogram
0470 END
0480
0490
```

end 109

(contd. from opposite page)


```
0500 REM COLLECTION SUBROUTINE
0510 MAT READ A(X)          target
0520 I=0                  } 'throw'
0530 I=I+1                }
0540 R=INT(K*RND+1)         select } subscript
0550 A(R)=A(R)-1            record }
0560 IF A$<>"NO" THEN MATPRINT A;          display current state on request
0570 IF A(5)=0 THEN A(5)=-1:A(1)=2:A(2)=2   if '5' obtained, 'reset', to collect 1^2, 2^2.
0580 FOR L=1 TO K:IF A(L)>0 THEN 530:NEXT L      check whether stage is complete
0590 IF N<11 THEN PRINT I;BS                print unit time, "stage" message, make N stage
0600 I1=I1+I                given length cumulate.
0610 IF I>100 THEN I=100:PRINT: PRINT "********OVERFLOW*********(";I;")"
                                    record each overflow value, but count I to display is unaffected.
0620 RETURN
0630
0640 DATA 0,0,0,0,-1,1
0650 DATA 0,0,4,1,1,0
0660
0670 REM HISTOGRAM SUBROUTINE
0680 REM FREQUENCIES HELD IN H(100), GROUPED IN K(100)
0690 PRINT "HISTOGRAM OF RESULTS"
0700 PRINT
0710
0720
0730
0740
0750 Y1=0:Y2=0              Σx  } accumulators of  Σf → mean
0760 Y0=0                   Σx  }                  Σf → variance
0770 FOR J=1 TO A1
0780 K(J)=0                 reset (grouped) freq. class
0790 FOR X=1 TO B1
0800 X=B1*(J-1)+X           select next X
0810 K(J)=K(J)+H(X)         increment freq. class
0820 Y0=Y0+H(X)             }
0830 Y1=Y1+H(X)*X           } increment  Σf
0840 Y2=Y2+X*X*H(X)         }            Σx.f
                                         Σx².f
0850 NEXT X
0860 PRINT (J-1)*B1+1;"TO";J*B1;TAB(12);   class boundaries
0870 B2=K(J)
0880 FOR L=1 TO B2:PRINT "*";:NEXT L    } output histogram
0890 PRINT
0900 NEXT J
0910 PRINT "MEAN = ";Y1/Y0                  }
0920 PRINT "S.D. = ";SQR(Y2/Y0-Y1*Y1/Y0/Y0)  } for histogram
0930 RETURN
```

Section 7.

Theoretical calculation of probabilities for some of the problems already encountered.

(Some of these examples assume familiarity with aspects of probability theory of about A-level standard).

Problem A (Rumour) (p 1).

(i) In a group of 6 people, (p4),

probability that only 2 people ever hear the rumour

= probability (process stops at stage 2) = $\frac{1}{5}$ = 0.2

probability (stops at 3)

= prob (does not stop at 2) x prob (choice of 2 knowledgable from 5)

= $\frac{4}{5} \times \frac{2}{5}$ = 0.32

prob (stops at 4) = $(\frac{4}{5} \times \frac{3}{5}) . \frac{3}{5}$ = 0.288

prob (stops at 5) = $(\frac{4}{5} . \frac{3}{5} . \frac{2}{5})$ = 0.1536

prob (stops at 6) = $(\frac{4}{5} . \frac{3}{5} . \frac{2}{5} . \frac{1}{5}) 1$ = 0.0384

(As a check, sum of probabilities = 1). Prob (stops at stage 1) = 0

Expected number of people hearing about the rumour

= 2 x 0.2 + 3 x 0.32 + 4 x 0.288 + 5 x 0.1536 + 6 x 0.0384 = 3.5104

Note that the probability that <u>all 6 will hear</u>, is small (0.04 or 4%)

(ii) In a group of 10 people, the probabilities of stopping at stage k are given by

k =	2	3	4	5	6	7	8	9	10
prob	$\frac{1}{9}$	$\frac{8.2}{9.9}$	$\frac{8.7.3}{9.9.9}$	$\frac{8.7.6.4}{9.9.9.9}$	$\frac{8.7.6.5.5}{9^5}$	$\frac{816}{319^6}$	$\frac{817}{2.9^7}$	$\frac{818}{9^8}$	$\frac{81}{9^8}$
	.11111	.19753	.23045	.20485	.14226	.07587	.02951	.00749	.00094

($\sum_{p7}$, [illegible]).

Again we may check $\sum$ probabilities = 1

Expected number hearing rumour = (2 x 0.11111 + 3 x 0.19753 + ...)

= 4.4583

(Computer simulation, 100 experiments, gave an average of 4.25). $\sum_{p196 \& p89}$.

Problem C (Christmas pudding - pp 15)

Probability that any given coin is in a one-twelfth portion of the pudding is $\frac{1}{12}$. For 12 coins we have a binomial distribution with $n = 12$, $p = \frac{1}{12}$ and $np = 1$.

This is approximately a Poisson distribution with mean $\lambda = 1$

x	0	1	2	3	4
prob(x coins in a portion)	$e^{-1} : 1$	$e^{-1} : 1$	$e^{-1} : \frac{1}{2}$	$e^{-1} : \frac{1}{6}$	$e^{-1} : \frac{1}{24}$
i.e.	0.368	0.368	0.184	0.061	0.015

So we expect $12 \times 0.368 \left(= 4.416 \text{ or } 4.5\right)$ portions to have no coins, and 4.5 portions to have 1, 2.2 to have 2, and about 1 to have 3 or more.

To make $P(0) = 5\%$ (say), we require to choose λ so that $e^{-\lambda} = \frac{1}{20}$, $\lambda = \log_e 20 \approx 3$ and so need about 36 coins.

Problem D

a) Waiting time to throw a '6' with a die

Prob(1st '6' on 1st throw) = $\frac{1}{6}$

Prob(1st '6' on 2nd throw) = prob(no '6' on 1st throw) x prob('6' on 2nd throw)

$= \frac{5}{6} \times \frac{1}{6}$

Prob(1st '6' on 3rd throw) = $\left(\frac{5}{6}\right)^2 \times \frac{1}{6}$

Hence expected wait for first '6' is

$1.\frac{1}{6} + 2.\frac{5}{6}.\frac{1}{6} + 3.\left(\frac{5}{6}\right)^2.\frac{1}{6} + \ldots.$

$= \frac{1}{6}.(1 + 2q + 3q^2 + \ldots.)$ where $q = \frac{5}{6}$

$= \frac{1}{6}.(1-q)^{-2} = 6$

(In general, if prob (A) = $\frac{1}{w}$, expected waiting time to 1st occurrence of A is w)

b) Collector's problem (p 1).

Suppose I have already collected 2 of the 6 badges. Then any one of the remaining 4 will count as 'success' if I obtain it next, i.e. p('success') = $\frac{4}{6}$

Thus expected waiting time to collect next badge is $\frac{6}{4}$. Generally if I have already collected $(6 - k)$ badges, so that k remain to be collected, the

expected wait for the next is $\frac{6}{k}$. Thus the total expected wait is

$$\frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 6(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}) = 14.7$$

For 10 badges, the expected wait is $10(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots\ldots\frac{1}{10}) = 29.3$

and for 20, it is $20(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots \frac{1}{20}) = 71.95$

(for large n, $n(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n})$ is about $n \log_e n$)

Computer simulation (100 experiments) produced average waits of 14.14, 30.58, 70.97 respectively.

Waiting Time for the Sequence HHH When a Coin is Tossed Repeatedly (see p 10).

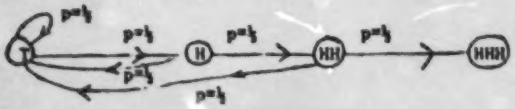
We can represent the possible states of affairs as in the diagram, where

T means last coin showed T

H means last two coins were TH

HH means last three coins were THH

HHH means last four coins were THHH



Denote the expected waiting times to the end of the game, from each state, by W_T, W_H, W_{HH} respectively.

Suppose we are in state HH. Then with probability ½ we move to state HHH (wait = 1), and with probability ½ we move to state T (wait = W_T + 1).

Thus $W_{HH} = \frac{1}{2}(W_T + 1) + \frac{1}{2}$. (1)

Similarly $W_T = \frac{1}{2}(W_T + 1) + \frac{1}{2}(W_H + 1)$

$W_H = \frac{1}{2}(W_T + 1) + \frac{1}{2}(W_{HH} + 1)$

from which $W_T = W_H + 2$

$2W_H = W_T + W_{HH} + 2$

$2W_{HH} = W_T + 2$

giving $W_T = 14$, $W_H = 12$, $W_{HH} = 8$ (cf. p 10 ~ p 117

Sum of Scores on 2, 3, 4 Dice (4.11).

2 Dice: Outcomes and resulting sum of scores are shown in the table. Since '4' can be obtained in 3 ways (as 1+3, 2+2, 3+1)

$$\text{prob (sum is 4)} = \frac{3}{36} = \frac{1}{12}$$

Hence theoretical probabilities are

s=	2	3	4	5	6	7	8	9	10	11	12
36p=	1	2	3	4	5	6	5	4	3	2	1

1st die \ 2nd die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

sum of scores

3 Dice: 3 can be obtained in 1 way (1 + 1 + 1)

4 can be obtained in 1+2 ways 2 + (1 + 1)

1 + (2 + 1)
or
1 + (1 + 2)

5 can be obtained in 1+2+3 ways ('first' die shows '3', '2', or '1')

.....

9 can be obtained in 2+3+4+5+6+5 ways (first die shows '6', '5', '4', '3', '2', '1')

10 can be obtained in 3+4+5+6+5+4 ways(" " " ")

.....

13 can be obtained in 6+5+4+3+2+1 ways(" " " ")

14 can be obtained in 5+4+3+2+1 ways (first die shows '6','5','4','3' or '2')

.....

17 can be obtained in 2+1 ways (first die shows '6' or '5', others give sum = 11 or 12)

18 can be obtained in 1 way (first die '6', others both '6')

Hence nos. of ways are:-

Score S	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
No. of ways (=216.p_s)	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1

(Where p_s = probability of a score of S)

Similarly for 4 Dice

Score	No. of Ways	
4	1	
5	1+3 -- ('2' on first and '3' on other 3, or '1' on first and '4' on other 3)	
6	1+3+6	
7	1+3+6+10	
8		
9	1+3+6+10+15+21	
10	3+6+10+15+21+25	
11	6+10+15+21+25+27	corresponding to first die
12	- - - - - - - - - - - -	= '6', '5', '4', '3', '2', '1'
13	- - - - - - - - -	
14	- - - - - -	
:	:	
:	:	
19	21+15+10+6+3+1	
20	15+10+6+3+1	first die = '6', '5', '4', '3', '2'
:	:	
:	:	
24	1	= '6'

Hence no. of ways are:-

Score s	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
No. of ways (=1296 x p_s)	1		10		35		80		125		146		125		80		35		10		1
		4		20		56		104		140		140		104		56		20		4	

(where p_s = probability of a score of s)

Number of Heads When N Coins are Tossed ($\S 11$).

The probability distributions are:-

N = 2

TT	(TH)	HH
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

N = 3

T^3	T^2	T	H^3
$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The outcome T^2 can occur as TTH, THT or HTT (in 3 ways). The distribution is Binomial with N=3, $p=\frac{1}{2}$, $(q=1-p)$

prob(no. of successes=K) $= {}_nC_k p^k q^{n-k}$

N = 4

No. of heads	0	1	2	3	4
prob	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

In the general case, the probability of K heads when N coins are tossed is

$$ {}_NC_K \left(\tfrac{1}{2}\right)^K \left(\tfrac{1}{2}\right)^{N-K} = \frac{{}_NC_K}{2^N} $$

e.g for N = 6, probabilities are $\frac{1}{64}, \frac{6}{64}, \frac{15}{64}, \frac{20}{64}, \frac{15}{64}, \frac{6}{64}, \frac{1}{64}$

Applying this to the problem of p27 (Christmas tree lights), where $p = \frac{1}{36}$, N = 10, the probabilities that 0,1,2, ... bulbs fail during a given year are

$$ \left(\tfrac{35}{36}\right)^{10}, \quad 10.\tfrac{1}{36}\left(\tfrac{35}{36}\right)^{9}, \quad \tfrac{10.9}{2}\left(\tfrac{1}{36}\right)^{2}\left(\tfrac{35}{36}\right)^{8}, \ldots $$

Length of Runs ($\S 12$)

Suppose a coin is tossed - the average waiting time to obtain 'T' is

$$ 1.\tfrac{1}{2} + 2.\tfrac{1}{4} + 3.\tfrac{1}{8} + 4.\tfrac{1}{16} + \ldots\ldots = \tfrac{1}{2}(1+2p+3p^2 \ldots.) $$

where $p = \frac{1}{2}$

$$ = \tfrac{1}{2}.\ \frac{1}{(1-p)^2} = 2 $$

So, the average length of a run of heads is 2 (and, equally, of a run of tails).

For a biased coin, prob (H) = p $\neq$ 0.5, prob (T) = q = 1-p

Average wait for H is $\frac{1}{p}$ and for T is $\frac{1}{q}$; so average length of a run of tails is $\frac{1}{p}$.

Now runs of H and of T must alternate, (so their number can differ by at most 1).

i.e. average run length $\simeq \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)$

e.g. 100 throws $p = q = 0.5$ No. of runs = 50

$p = 0.6,\ q = 0.4$ No. of runs $= \frac{200}{\frac{1}{0.6} + \frac{1}{0.4}} = \frac{40}{48}$

$p = 0.8,\ q = 0.2$ $N = \frac{200}{\frac{1}{0.8} + \frac{1}{0.2}} = 32$

$p = 0.9,\ q = 0.1$ $N = 18$

(as $p \to 1,\ q \to 0,\ N \to 0$)

For dependent events

$$\text{1st} \begin{cases} 1 \\ 2 \end{cases} \overbrace{\begin{pmatrix} 1 & 2 \\ p & 1-p \\ 1-q & q \end{pmatrix}}^{\text{2nd}}$$

On '2', average wait for '1' is $\frac{1}{1-q}$

On '1' average wait for '2' is $\frac{1}{1-p}$

} so average run length $= \frac{1}{2}\left(\frac{1}{1-q} + \frac{1}{1-p}\right)$

e.g. 100 throws, $p = q = 0.9$, (c.f. 'staying put')

$$N = \frac{200}{10+10} = 10$$

and 100 throws, $p = q = 0.1$, ("flip-flop")

$$N = \frac{200}{\frac{1}{0.9} + \frac{1}{0.9}} = 90$$

(max. no. of runs would be 100)

Thus bias tends to reduce no. of runs in independent trials; dependence tends to reduce no. of runs, if 'positive', and increase no. of runs if 'negative'. (cf. p 12).

Dalekium (*p. 20*).

If we treat this as a deterministic, continuous process, the appropriate model is $\frac{dx}{dt} = -kx$, where x = amount of Da in g , t = time in min, and k is a constant. (*cf.* problem (E) p. 2). From this, $y = Ae^{-kt}$, where A is the initial amount of Da and k is given by $e^{-k.10} = \frac{1}{2}$

i.e. $10k = \log_e 2$ or $k = 0.069315$

Then since $e^{k} = 2^{\frac{1}{10}}$, we may write $\underline{x = A.2^{-t/10}}$

Alternatively using a stochastic model, the expected number of coins remaining after 1 throw is $\frac{1}{2} \times 100$, after 2 throws $(\frac{1}{2})^2 \times 100$, etc.

$N = 100 \times 2^{-T}$ where $T = \frac{t}{10}$ = time measured in 10 min. periods.

(Proof

For any given coin, probability it will survive T throws is $(\frac{1}{2})^T$.

Considering the group of 100 coins we have a Binomial experiment with $p = (\frac{1}{2})^T$, $n = 100$, and the expected number of 'successes' is $np = 100(\frac{1}{2})^T$.)

Bacteria (*p. 26*).

The probability distribution of bacteria among samples will be a Poisson distribution; suppose it has mean λ i.e.

Prob. (number of bacteria in a sample is k) $= e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

(Then E(total number) $= 1.e^{-\lambda}.\lambda + 2.e^{-\lambda}.\frac{\lambda^2}{2!} + 3.e^{-\lambda}.\frac{\lambda^3}{3!} + \ldots = e^{-\lambda}.\lambda.\frac{d}{d\lambda}(e^{\lambda}) = e^{-\lambda}.\lambda.e^{\lambda}$
$= \lambda$ as required.)

In particular, Prob. (no bacteria in sample) $= e^{-\lambda}.1 = P$, say.

If we can estimate P (and so λ), the estimated total number of bacteria is 100λ (since λ = mean number per sample).

But expected number of uncontaminated samples is 100P, so that the estimate of P is given by $100P = 50$, $P = \frac{1}{2}$. $\lambda = \log_e 2 = 0.6931471$

Estimated total no. of bacteria = $100\lambda = 69.3$

Waiting Time Problems - e.g. waiting time to throw both a '5' and a '6' on a die.† (See 'Beetle', p 29.).

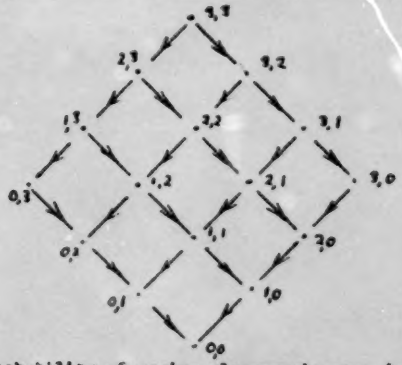
A general problem is to find the expected wait to achieve a target of a '5's and b '6's, denoted by the "target vector" (a,b). (See programs FIV6, COLLGEN, pp 42-3)

Particular cases - to throw a single '6' i.e. (0,1) (see also p 107).

to throw 2 '6's i.e. (0,2)

to throw at least 4 of each i.e. (4,4) - cf. 'Octopus' (p.29).

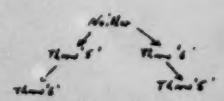
Consider the problem (3,3) and denote the intermediate states as in the diagram, where (a,b) means that a '5's' and b '6's' have yet to be collected. (Note the change of notation)



The probability of moving along each arrow is $\frac{1}{6}$

† Note also the 'tree diagram' for 5.6 is

(the two final states — to be identified).



Suppose we are at say, (2,1) and let the expected waiting time to reach (0,0) be W(2,1). On the next throw we stay there with probability $\frac{2}{3}$, and move on with probability $\frac{1}{3}$.

If we move, our expected waiting time is $\frac{1}{2}(W_{(2,0)} + W_{(1,1)})$ - since both outcomes are equally likely.

Thus we have $W_{(2,1)} = 3 + \frac{1}{2}\left(W_{(2,0)} + W_{(1,1)}\right)$ .....(I)

wait to 'move' ⊕ wait when 'move' achieved

If the target is to throw a single 'six', the expected wait is 6 throws

For if n throws are required, (compare p 107),

$$E(n) = 1.\frac{1}{6} + 2.\frac{5}{6}.\frac{1}{6} + 3(\frac{5}{6})^2.\frac{1}{6} + 4(\frac{5}{6})^3.\frac{1}{6} + \cdots$$

$$= p(1 + 2q + 3q^2 + \cdots\cdots) \qquad \text{where } q = \frac{5}{6} = 1-p$$

$$= p.\frac{1}{(1-q)^2} = p.\frac{1}{p^2} = \frac{1}{p} \quad (=6)$$

If we measure waiting time in units of 6

(so that "wait for a single '6'" = 1 unit),

the equation (I), above becomes

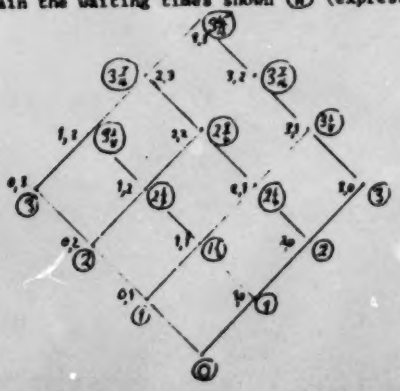
$$W_{(2,1)} = \frac{1}{2} + \frac{1}{2}\left(W_{(2,0)} + W_{(1,1)}\right) \qquad \cdots\cdots (II)$$

Similarly, considering the position (0,2)

$$W_{(0,2)} = 1 + W_{(0,1)} \cdots\cdots\cdots (III)$$

(to 'move')

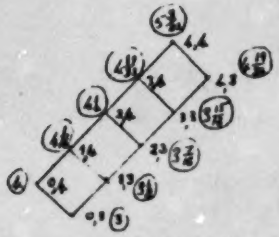
Using similar equations, and working back through the network from (3,3) we obtain the waiting times shown Ⓦ (expressed in units of 6)



$W_{11} = \frac{1}{2} + \frac{1}{2}(1+1)$

$= 1\frac{1}{2}$

Clearly the diagram can be extended - e.g. to find the wait to reach (4,4) we add further positions, and use similar equations.

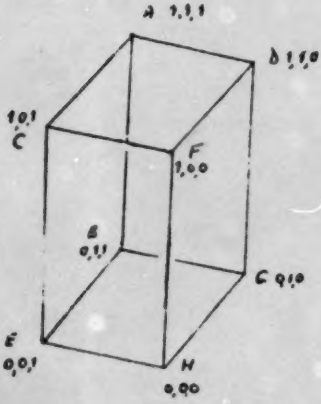


From this diagram we read off waiting times:

For (0,2)	2 units (i.e. 12 throws)
For (0,3)	3 units (i.e. 18 throws)
For (1,2)	$2\frac{1}{2}$ units (i.e. $13\frac{1}{2}$ throws)
For (1,3)	$3\frac{1}{8}$ units (i.e. $18\frac{3}{4}$ throws)
For (4,4)	$5\frac{3}{32}$ units (i.e. $30\frac{9}{16}$ throws: 30.6)
	(c.f. p 27 , "Octopus")

A similar method deals with more complex problems e.g. waiting time for all three of '4', '5' and '6' - i.e. target (1,1,1) (p 137 and contrast p 78).

The diagram is three-dimensional:- (i,j,k) denotes $\left\{\begin{array}{ll} i & \text{'4's} \\ j & \text{'5's} \\ k & \text{'6's} \end{array}\right\}$ to be obtained



$W_E = W_F = W_G = 1$ unit (= 6 throws)

$W_B = \frac{1+1}{2} + 1 = 1\frac{1}{2} = W_C = W_D$

$W_A = \frac{1\frac{1}{2} + 1\frac{1}{2} + 1\frac{1}{2}}{3} + 1 = 1\frac{5}{6}$ or 11 throws

wait after 'move' ↗ ↖ wait to 'move'

If instead of using a die, we were collecting random digits $\emptyset$ ..... 9, with target '4', '5' and '6', unit waiting time would be 10 'throws', and the equations would read:-

$W_E = W_F + W_G = 1$ unit (= 10 "throws")

$W_B = \frac{1+1}{2} + \frac{1}{2} = 1\frac{1}{2}$ units (since the expected 'wait to move' is now 5 throws: $p = \frac{1}{5}$

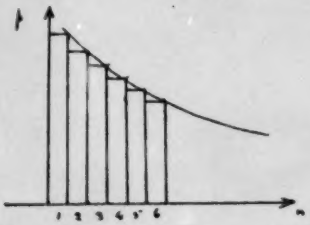
i.e. results are as before in 'units'

Thus $W_A = \frac{11}{6} \times 10 = 18\frac{1}{3}$ "throws"

Lucky 7 (see p 30)

An icosahedral die is thrown repeatedly until a '7' appears. (There are 20 faces numbered in pairs with the digits $\emptyset$ .... 9, so that $p(7) = \frac{1}{10}$, as for random digits). We seek the distribution of waiting times and the expected waiting time. The latter is 10 throws, (by analogy with 'throwing a "6" on a cubical die', $W = \frac{1}{p} = \frac{1}{0.1} = 10$), but we also calculate it from the distribution.

Denote (first '7' on throw n) by S_n



Then $P(S_1) = \frac{1}{10}$

$P(S_2) = \frac{9}{10} \cdot \frac{1}{10}$

$P(S_3) = (\frac{9}{10})^2 \cdot \frac{1}{10}$

.....

The distribution is as shown and the expected waiting time is

$E(W) = 1.\frac{1}{10} + 2.\frac{9}{10}\ \frac{1}{10} + 3(\frac{9}{10})^2 \cdot \frac{1}{10}$ .....

$= \frac{1}{10}(1 + 2q + 3q^2 + ...)$, where $q = \frac{9}{10}$

$= \frac{1}{10}\ (\frac{1}{1-q})^2 = 10$ as above

The distribution of waiting times is similar to the graph showing radioactive decay with time (see p. 20, and below). Repeated tossing of a coin or die may be used to simulate the decay process, where the dependence on time is shown by an almost identical graph - (each is a 'geometric' distribution, though the underlying reasons are not exactly the same).

Prob. (wait ≤ 2) is $\frac{1}{10} + \frac{9}{10} \cdot \frac{1}{10} = 0.19$ ("lucky")

Prob. (wait ≥ 16) is $1 - \frac{1}{10}\left(1 + \frac{9}{10} + (\frac{9}{10})^2 \ldots\ldots (\frac{9}{10})^{15}\right)$

$$= 1 - \frac{1}{10}\left(\frac{1 - (\frac{9}{10})^{16}}{\frac{1}{10}}\right) = (\frac{9}{10})^{16} = 0.1853 \qquad \text{("unlucky")}$$

Computer simulation of this process produced the following results (2 sets of 100 trials each).

A)

Wait	1	2	≥ 16
Freq.	10	13	13
	23		

B)

1	2	≥ 16
12	5	24
17		

Problem E. (Two-stage reaction, p 2).

Consider first a simplified version - a one-stage reaction (cf. simple radioactive decay) in which a substance A changes into B, the rate of transformation being proportional to the amount of A present.

Let the amount of A be a initially, and u at any subsequent time and let

$\frac{du}{dt} = -k_1 u$ Then, integrating, $u = Ae^{-k_1 t}$

where A is a constant. Since $u = a$ when $t = 0$, we have $u = ae^{-k_1 t}$

Now consider the two stage reaction $A \longrightarrow B \longrightarrow C$

amount present at time t	u	v	w

We have

$$\frac{du}{dt} = -k_1 u \quad , \qquad \frac{dv}{dt} = -k_2 v + k_1 u \quad , \qquad \frac{dw}{dt} = k_2 v$$

Hence $u = ae^{-k_1 t}$ as before, and then $\frac{dv}{dt} = -k_2 v + k_1 a\, e^{-k_1 t}$

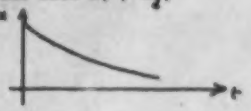
which gives
$$\begin{cases} v = \frac{ak_1}{k_2 - k_1} e^{-k_1 t} + M e^{-k_2 t} & (k_1 \neq k_2) \\ \text{or} \\ v = a k_2 t\, e^{-k_2 t} + M e^{-k_2 t} & (k_1 = k_2) \end{cases}$$

where M is an arbitrary constant.

[The equations may be solved by the standard method of multiplying throughout by an integrating factor, $e^{k_2 t}$, but the correctness of the solution can be checked merely by differentiation].

Using the initial condition $v(0) = 0$ we have, in the case $k_1 \neq k_2$,

$$v = \frac{ak_1}{(k_2 - k_1)} \left(e^{-k_1 t} - e^{-k_2 t} \right)$$



and if $k_1 = k_2$

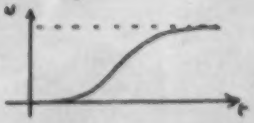
$$v = ak_2 t e^{-k_2 t}$$

From these results, using $\frac{dw}{dt} = k_2 v$, we obtain:—

$$w = a\left(1 + \frac{k_1}{k_2 - k_1} e^{-k_2 t} - \frac{k_2}{k_2 - k_1} e^{-k_1 t}\right) \qquad (k_1 \neq k_2)$$

or $\quad w = a\left(1 - e^{-k_2 t} - k_2 t e^{-k_2 t}\right) = a\left(1 - \frac{1 + k_2 t}{e^{k_2 t}}\right)$,

for $k_1 = k_2$



The maximum value of v occurs when $\frac{dv}{dt} = 0$
For $k_1 \neq k_2$, this is when $-k_1 e^{-k_1 t} = -k_2 e^{-k_2 t}$

$$\Rightarrow t = \frac{1}{k_2 - k_1} \log_e \left(\frac{k_2}{k_1}\right).$$

and then $V = \frac{ak_1}{k_2 - k_1} \left\{ \left(\frac{k_1}{k_2}\right)^{\frac{k_1}{k_2 - k_1}} - \left(\frac{k_1}{k_2}\right)^{\frac{k_2}{k_2 - k_1}} \right\}$

For $k_1 = k_2$, we have, at the stationary point,

$$t. -k_2 e^{-k_2 t} + e^{-k_2 t}. 1. = 0$$

$$t = \frac{1}{k_2}$$

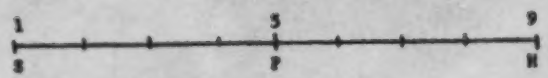
$$\underline{V = a/e}$$

In particular, if $k_1 = 2k_2$, $\quad V \max = -2a. \left((\tfrac{1}{2})^2 - (\tfrac{1}{2})\right) = 2a\,(\tfrac{1}{2} - \tfrac{1}{4}) = \frac{a}{2}$

and if $k_1 = \frac{1}{2}k_2$, $\quad V \max = a. \left((\tfrac{1}{2})^1 - (\tfrac{1}{2})^2\right) = a.\tfrac{1}{4} = \frac{a}{4}$

Random Walk on a Line (2 problems)

1. (c.f. p 26.)



Consider the drunkard who starts at P and moves one unit of distance to the right with probability p, and one unit to the left with probability q. Let U_k be the probability he reaches home from position k.

$U_9 = 1$, $U_1 = 0$, then $U_k = pU_{k+1} + qU_{k-1}$ (for $k = 2,3,\ldots,8$)

Suppose $U_k = \lambda\alpha^k$, then α satisfies $\alpha = p\alpha^2 + q$

One root of this equation is 1, since $p + q = 1$

The other root is $\alpha = \frac{1-p}{p} = \frac{q}{p} - 1$, and is positive.

The general solution is $U_k = A\alpha^k + B.1^k$ where A and B are constants,

or, equivalently: $U_k = A'\alpha^{k-1} + B$ where $A' = A\alpha$

Using $\left.\begin{array}{l} U_9 = 1, \quad A'\alpha^8 + B = 1 \\ U_1 = 0, \quad A' + B = 0 \end{array}\right\}$ $A' = \frac{1}{\alpha^8 - 1}$, so that $U_k = \frac{\alpha^{k-1} - 1}{\alpha^8 - 1}$

In particular if he starts at position 5,

$U_5 = \frac{\alpha^4 - 1}{\alpha^8 - 1} = \frac{1}{\alpha^4 + 1}$ and if $p = \frac{1}{2}$, $\alpha = 1$, $U_5 = \frac{1}{2}$ as expected.

Values of U_5 may be tabulated:-

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7
U_5	1.9×10^{-4}	3.9×10^{-3}	3.26×10^{-2}	0.1649	0.5	0.8351	0.9674

(N.B. Considering the interchange of p,q, and of '1' and '9', we see that the values of U_5 for p = 0.2 and p = 0.8 sum to 1). (See Aspinall and Hopcroft (1968)).

For an unbounded random walk on a line, (no absorbing states), it may be shown that the average distance from the origin after n steps is proportional to $\sqrt{n}$. See Dynkin and Uspenskii (1933). The program WALK (p85) investigates this empirically.

2. (A slightly different problem)

A drunkard sets out to walk home. In each successive unit of time, he has a chance $p, > \emptyset$, of walking 1 unit E, a chance $q, > \emptyset$, of walking 1 unit W, and a chance $(1 - p - q)$ of going to sleep where he is, in which case the process stops. His home lies n units E of his starting point, where $n > 0$; once he gets home he stays there. Let c_n be his chance of getting there before he goes to sleep. Find c_n.

$\therefore C_n = qC_{n+1} + pC_{n-1}$, $n \geq 1$, with $C_0 = 1$

Let $C_n = \lambda \alpha^n$, then $q\alpha^2 - \alpha + p = 0$

Thus, $C_n = A\alpha^n + B\beta^n$ where α, β are the roots of $qt^2 - t + p = 0$

But since $f(1) = q - 1 + p < \emptyset$, one of the roots is greater than 1, and if this is β, we must have $B = 0$ (since $C_n \to \emptyset$ as $n \to \infty$)

Hence $C_n = A\alpha^n$ and using $C_0 = 1$, $A = 1$

i.e. $C_n = \alpha^n$ where α is the smaller root of $qt^2 - t + p = 0$

Birthday Problem (p.70).

Suppose we first ignore the effect of leap years:

For 2 people, probability of no coincidence of birthdays is $\frac{364}{365}$

For 3 " " " " " " " $\frac{364}{365} \cdot \frac{363}{365}$

For (N+1) " " " " " " " $\frac{364.363. \ldots .(363-N)}{365^N}$

The probability that at least two birthdays coincide in a group of (N+1) people, is

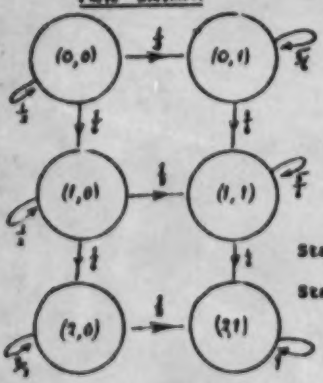
$1 - \frac{364.363. \ldots .(365-N)}{365^N}$; $\beta > 1$ for $N \geq 22$. (Cf. p 70).

Section 8.

Markov Chains

On pp 29, 116 we considered the game of 'Octopus' in which the object is to roll a die until four 1's and four 2's are obtained.

Here we consider by way of illustration an even simpler situation. A die is to be rolled until two 1's and any other odd number, in any order, are obtained. The game may be represented by the diagram showing the six possible states, where (x,y) means I have obtained x 1's and y other odd numbers. The game ends when x⩾2 and y⩾1. (There is clearly no need to record values of x>2 or of y>1). (1)



(Arrows are labelled with transition probabilities)

There are six possible states and we can represent the probabilities of transition between states by the matrix P, the matrix of transition probabilities, where P_{ij} is the probability of a transition starting in state i and finishing in state j.

Transition matrix

Starting State \ Finishing State	(0,0)	(1,0)	(0,1)	(2,0)	(1,1)	(2,1)
(0,0)	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	0	0	0
(1,0)	0	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{3}$	0
(0,1)	0	0	$\frac{5}{6}$	0	$\frac{1}{6}$	0
(2,0)	0	0	0	$\frac{2}{3}$	0	$\frac{1}{3}$
(1,1)	0	0	0	0	$\frac{5}{6}$	$\frac{1}{6}$
(2,1)	0	0	0	0	0	1

$= P$

(1) A notation different from that of pp. 116, 118, when it is convenient to avoid 'complements'.

A process in which the probability of transition from one state to another depends only on the starting and finishing states, as in this example, is called a Markov Chain Process *

(* A.A. Markov, 1856-1922, a Russian mathematician)

The probability of a move from state (0,0) to state (1,0) is $\frac{1}{6}$, and the probability of being in each of the states after one step, starting in state (0,0), is given by the vector $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}, 0,0,0)$, which is the first row of P.

What can we say about the situation after 2 steps from (0,0)? For example, the probability of being in state (2,0) is $\frac{1}{36}$, (from the diagram above), and the probability of being in state (1,1) is $\frac{1}{6} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{6} = \frac{1}{9}$. Considering the probabilities for all states after two transitions from (0,0), we have $(\frac{1}{4}, \frac{1}{9}, \frac{4}{9}, \frac{1}{36}, \frac{1}{6}, 0)$
obtained as $(\frac{1}{2} \times \frac{1}{2}, \frac{1}{2} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{2}, \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{5}{6}, \frac{1}{6} \times \frac{1}{6}, \frac{1}{6} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{6}, 0)$
(As a check we note that the sum of the probabilities is $\frac{9 + 6 + 16 + 1 + 4}{36} = 1$)

If we consider this in matrix terms, starting from the state (0,0) - represented by the probability vector (1,0,0,0,0,0) - the probabilities after one step are represented by the vector

$$(1\ 0\ 0\ 0\ 0\ 0) \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{5}{6} & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (\tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{3}, 0,0,0)$$
i.e. the first row of P

and after two steps by

$$(\tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{3}, 0,0,0) \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{5}{6} & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (1\ 0\ 0\ 0\ 0\ 0)\, P^2$$
i.e. the first row of P^2

We note that the probability of transition from state i, via some state k, to state j, is $p_{ij}\ p_{ij} + p_{i2}\,p_{2j} + p_{i3}\,p_{3j} + \cdots\cdots$

$$= \sum_k p_{ik}\,p_{kj} = (i,j)\text{ th element of } P^2$$

Similarly, the probability vector for the states after 3 steps is

$$(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{4}{9}, \tfrac{1}{36}, \tfrac{1}{9}, 0)P = (\tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{3}, 0,0,0)P^2 = (1,0,0,0,0,0)P^3$$

$$= \text{first row of } P^3$$

Consequently, the (i,j) th element of the matrix P^n gives the probability for transitions from state i to state j in n steps.

Example In the case considered

$$P = \frac{1}{36}\begin{pmatrix} 36 & 1 & 2 & 0 & 0 & 0 \\ 0 & 36 & 0 & 1 & 2 & 0 \\ 0 & 0 & 36 & 0 & 1 & 0 \\ 0 & 0 & 0 & 36 & 0 & 2 \\ 0 & 0 & 0 & 0 & 36 & 1 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix} \qquad P^2 = \frac{1}{36}\begin{pmatrix} 36 & 4 & 16 & 1 & 4 & 0 \\ 0 & 36 & 0 & 7 & 16 & 4 \\ 0 & 0 & 25 & 0 & 10 & 1 \\ 0 & 0 & 0 & 36 & 0 & 20 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

i.e.

$$P^2 = \begin{pmatrix} .2500 & .1667 & .4444 & .0278 & .1111 & 0 \\ 0 & .2500 & 0 & .1944 & .4444 & .1111 \\ 0 & 0 & .6944 & 0 & .2778 & .0278 \\ 0 & 0 & 0 & .1975 & 0 & .8025 \\ 0 & 0 & 0 & 0 & .4823 & .3177 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad P^4 = \begin{pmatrix} .0625 & .0535 & .4398 & .0517 & .1615 & .0102 \\ 0 & .0625 & 0 & .1950 & .4197 & .3317 \\ 0 & 0 & .4823 & 0 & .3858 & .1319 \\ 0 & 0 & 0 & .1975 & 0 & .8025 \\ 0 & 0 & 0 & 0 & .4823 & .5177 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^8 = \begin{pmatrix} .0039 & .0104 & .2287 & .0247 & .3617 & .3706 \\ 0 & .0039 & 0 & .0351 & .2287 & .7323 \\ 0 & 0 & .2326 & 0 & .3721 & .3953 \\ 0 & 0 & 0 & .0390 & 0 & .9610 \\ 0 & 0 & 0 & 0 & .2326 & .7674 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad P^{16} = \begin{pmatrix} .00002 & .00006 & .0541 & .0014 & .1731 & .7714 \\ 0 & .00002 & 0 & .0013 & .0541 & .9444 \\ 0 & 0 & .0541 & 0 & .1731 & .7728 \\ 0 & 0 & 0 & .0013 & 0 & .9985 \\ 0 & 0 & 0 & 0 & .0541 & .9459 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Notice that state 6 once entered, is never left - it is an absorbing state. The other states 1 - 5 are transient states. It is intuitively obvious, and may be demonstrated rigorously, that after a large number of transitions, the process will almost certainly be in state 6 (i.e with probability $\longrightarrow 1$ as number of transitions $\longrightarrow \infty$). The matrix powers P^n appear to be approaching the limit:- $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. As further examples:-

$P^{12} = \begin{pmatrix} & & \cdot 02 & \cdot 98 \\ & Q & 0 & 1 \\ & & \cdot 02 & \cdot 98 \\ & & 0 & 1 \\ & & 0 & 1 \\ & & 0 & 1 \end{pmatrix}$ and $P^{44} = \begin{pmatrix} & & \cdot 0001 & \cdot 9999 \\ & Q & 0 & 1 \\ & & \cdot 0001 & \cdot 9999 \\ & & 0 & 1 \\ & & 0 & 1 \\ & & 0 & 1 \end{pmatrix}$ (where Q is a 6×6 matrix of zeros)

We now consider three questions.

i) How long, on average, is spent in each of the transient states?

ii) How long, on average, elapses before the process reaches an absorbing state? ('mean free time to absorption')?

iii) If there is more than one absorbing state, what are the respective probabilities of absorption in each?

The answer to (i) will depend on the starting state - e.g. in the example above, if we start in state 2 i.e. (1,0), we can never re-enter state 1 i.e. (0,0). A little thought shows that the answer to (ii) follows quickly from (i) - for the sum of the number of steps spent in each transient state gives the mean free time to absorption.

Returning to the example above, suppose we start in state 1 i.e. (0,0)

After 0 steps we are in state 1 with probability 1

After 1 step we are in state 1 with probability p_{11} $(= \frac{1}{2})$

After 2 steps we are in state 1 with probability $p_{11}^{(2)}$ $(= \frac{1}{4})$

After 3 steps we are in state 1 with probability $p_{11}^{(3)}$ $(= \frac{1}{8})$

etc

(where $p_{ij}^{(k)}$ is the (i,j)th element of the matrix P^k)

So the mean number of steps spent in state 1 is

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots$ (=2, using the 'sum to infinity' of a geometric series)

In symbols this is $1 + p_{11} + p_{11}^{(2)} + p_{11}^{(3)} + \ldots\ldots\ldots$

Similarly after 0 steps we are in state 2 with probability 0

1	=	2	=	p_{12} $(=\frac{1}{2})$
2	=	2	=	$p_{12}^{(2)}$ $(=\frac{1}{4})$
3	=	2	=	$p_{12}^{(3)}$ $(=\frac{1}{8})$

etc.

Mean number of steps in state 2 is $0 + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ldots$

$= 0 + p_{12} + p_{12}^{(2)} + p_{12}^{(3)} + p_{12}^{(4)} + \ldots\ldots\ldots\ldots + p_{12}^{(n)} + \ldots\ldots\ldots\ldots$

$= (I + P + P^2 + P^3 \ldots\ldots\ldots + \ldots\ldots + P^n \ldots\ldots)_{12}$

In general the mean number of steps spent in state j, starting in state i, is the (i,j)th element of the infinite sum matrix

$(I + P + P^2 + P^3 + P^4 + \ldots\ldots\ldots)$

We are interested only in what happens in the non-absorbing states - in this case, states 1 - 5. We may write the matrix P as $\left(\begin{array}{c|c} Q & R \\ \hline S & I \end{array}\right)$

where Q is $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$, R is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \\ \frac{1}{4} \end{pmatrix}$, and S is $(0\ 0\ 0\ 0\ 0)$.

This is called displaying the matrix in "partitioned" form (or "block" form).

It may be shown either by considering the method for multiplication of partitioned matrices, or by repeating the above argument in terms of Q (instead of P), that, providing $i, j \leq 5$, the (i,j)th elements of $(I + P + P^2 + P^3 + P^4 \ldots)$ and $(I + Q + Q^2 + Q^3 + Q^4 + Q^5 \ldots.)$ are identical.

Since Q involves only transient states, the matrix Q^n must be 'close to' the zero matrix

$$O = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and, by analogy with

$1 + x + x^2 + x^3 + x^4 + x^5 + \ldots\ldots = \frac{1}{(1-x)} = (1-x)^{-1}$,

we will assume $(I + Q + Q^2 + Q^3 + \ldots) = (I - Q)^{-1}$.

For convenience the matrix $(I - Q)^{-1}$ will be called N.

Provided each element of Q^n is vanishingly small for large n, we can argue as follows $(I + Q + Q^2 + \ldots\ldots + Q^{n-1})(I - Q) = I - Q^n \to I$,

so that $(I + Q + Q^2 + Q^3 + \ldots + Q^{n-1})$ "approaches" $N = (I - Q)^{-1}$ as $n \longrightarrow \infty$

Thus we arrive at the solution to problem (i), p 126 :-.

The mean time spent in each non-absorbing state j, starting in state i, is the (i,j)th element of $(I - Q)^{-1}$, i.e. of N.

It also follows (see problem(ii)), p 127:-

The mean time to absorption, starting in state i, is the sum of the elements of the ith row of N. (For it is $n_{i1} + n_{i2} + n_{i3} + \ldots\ldots$ i.e. $\sum_{k} n_{ik}$ summed over all non-absorbing states k)

In the example above, $(I - Q)$ is $\frac{1}{6}\begin{pmatrix} 3 & -1 & -2 & 0 & 0 \\ 0 & 3 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and N is $6\begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{1}{12} & \frac{5}{12} \\ 0 & \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

Starting in state 1, the mean number of steps spent in each state is 2, $\frac{2}{3}$, 4, $\frac{1}{2}$, $5\frac{1}{2}$, and the mean number of steps to absorption is $12\frac{1}{2}$.

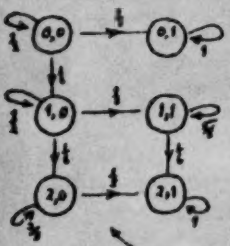
These results may be compared with the results of two simulations (see p 76) for which the means, each based on 100 trials were $\begin{cases} 2.18 & 0.66 & 3.97 & 0.53 & 5.39 \\ 2.28 & 0.65 & 2.95 & 0.26 & 4.63 \end{cases}$ steps, in each of the states 1 - 5 respectively, and the mean number of steps to absorption was $\begin{cases} 12.73 \\ 10.77 \end{cases}$

Another Example

Suppose we change, very slightly, the rules of the game given above on p 123, as follows:- [If '3' appears first, the chick is walking, otherwise it is running].

A die is rolled. If a '3' or a '5' appears before a '1', stop at once. If a '1' appears before a '3' or a '5', stop when a total of two '1's and at least one other odd number have appeared. (see DIAGRAM 2, p 75).

The diagram for the alternative game and the associated probability matrix are:-



Starting State	Finishing State (0,0)	(1,0)	(0,1)	(2,0)	(1,1)	(2,1)
(0,0)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	0	0	0
(1,0)	0	$\frac{1}{3}$	0	$\frac{1}{6}$	$\frac{1}{3}$	0
(0,1)	0	0	1	0	0	0
(2,0)	0	0	0	$\frac{1}{3}$	0	$\frac{1}{3}$
(1,1)	0	0	0	0	$\frac{5}{6}$	$\frac{1}{6}$
(2,1)	0	0	0	0	0	1

$= P$

(Arrows are labelled with transition probabilities)

As before, states may be denoted by pairs (x,y) indicating that x '1's' and y other odd numbers have appeared - and again we do not record values of x, y greater than 2 and 1 respectively.

The state denoted by (2,0) has now become an absorbing state and it is convenient to re-arrange the state labels thus:-

$$\begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 3 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 4 & 5 & 3 & 6 \end{array} \\ \begin{pmatrix} \frac{1}{4} & \frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{5}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

As in the example above, $P = \left(\begin{array}{c|c} Q & R \\ \hline S & I \end{array}\right)$

where $S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $R = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{6} \end{pmatrix}.$$

$$I-Q = \frac{1}{6}\begin{pmatrix} 3 & -1 & 0 & 0 \\ 0 & 3 & -1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ and N is $6\begin{pmatrix} \frac{1}{3} & \frac{1}{9} & \frac{1}{18} & \frac{2}{9} \\ 0 & \frac{1}{3} & \frac{1}{6} & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

so finally we have:-

Mean time, starting in state 1, $\begin{cases} \text{spent in state 1 is} & 2 \\ \text{spent in state 2 is} & \frac{2}{3} \\ \text{spent in state 4 is} & \frac{1}{3} \\ \text{spent in state 5 is} & 1\frac{1}{3} \end{cases}$

and mean time to absorption is $4\frac{1}{3}$.

(Compare with p 157 ; the corresponding results for a simulation (mean of 100 trials) were 1.88, 0.73, 0.33, 1.33 and 4.27).

There are now <u>two</u> absorbing states, and starting in state 1, the process is evidently more likely to be absorbed in state 3 than in state 6.

<u>Problem (iii)</u>

Find, for each starting state, the probabilities of absorption into states 3 and 6 respectively.

Starting in state 1, suppose the process passes through a succession of non-absorbing states, ending in state k, and then is absorbed by the transition $k \longrightarrow 3$.

The probability of absorption into state 3 is

$$r_{13} + \sum_k q_{1k} \cdot r_{k3} + \sum_k q_{1k}^{(2)} \cdot r_{k3} + \sum_k q_{1k}^{(3)} \cdot r_{k3} + \cdots\cdots$$

(after 1 step), (after 2 steps), (after 3 steps)

which is the element (1,3) of $(I + Q + Q^2 + Q^3 + Q^4 + \cdots\cdots)R$.

Similarly the probability of absorption, starting in state 1, into state 6, is the (1,6)th element of $(I - Q)^{-1}.R$ i.e. of NR.

In general, the probability of being absorbed in state j, from starting state i, is the (i,j)th element of NR.

In example (i) above, R is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{6} \end{pmatrix}$ NR = $6\begin{pmatrix} \frac{1}{24} + \frac{4}{27} \\ \frac{1}{18} + \frac{1}{9} \\ 0 + \frac{1}{6} \\ \frac{1}{6} + 0 \\ 0 + \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

i.e. the probability is 1 in each case.

In example (ii) above, R is $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{6} \end{pmatrix}$, NR is $6\begin{pmatrix} \frac{1}{9} & \frac{1}{24} + \frac{1}{27} \\ 0 & \frac{1}{18} + \frac{1}{9} \\ 0 & \frac{1}{6} + 0 \\ 0 & 0 + \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$

i.e. the probability is 1 from states 2,4,5, and $\frac{2}{3}$ or $\frac{1}{3}$ from state 1

agreeing with our intuitive judgement above.

(See p. 146, 157) a computer simulation of 100 trials produced the following results. (Absorbed in state 3, 66 times, absorbed in state 6 34 times. Mean numbers of steps to absorption, from each possible starting state, were 4.27, 7.11, 2.83, 5.21).

Examples of some other problems which may be approached in the same way are given below:

Example 1. (See p. 114).

(a) Waiting time to throw both a '5' and a '6' on a die :-

Denote states by (0,0) = state 1, (1,0) = state 2, (0,1) = state 3, (1,1) = state 4 (absorbing)

The matrices are

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & \frac{5}{6} & 0 & \frac{1}{6} \\ 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad Q = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{5}{6} & 0 \\ 0 & 0 & \frac{5}{6} \end{pmatrix} \qquad I-Q = \begin{pmatrix} \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$N = 6\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and row sums are } \begin{pmatrix} 9 \\ 6 \\ 6 \end{pmatrix}$$

(Mean no of steps to throw a '5' and a '6' is 9)

(b) To throw a '6' alone.

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ 0 & 1 \end{pmatrix} \end{array} \qquad Q = \left(\frac{5}{6}\right) \qquad I - Q = \left(\frac{1}{6}\right) \qquad N = (6)$$

Mean waiting time for a '6' is 6 steps. (cf. p. 107, and cf. also, above, where, given that a '5' has already been thrown (i.e. state 2), mean time to absorption in state 4 is 6 steps).

(c) To throw <u>two</u> '6's':-

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left(\begin{array}{cc:c} \frac{5}{6} & \frac{1}{6} & 0 \\ 0 & \frac{5}{6} & \frac{1}{6} \\ \hdashline 0 & 0 & 1 \end{array}\right) \end{array} \qquad I - Q = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ 0 & \frac{1}{6} \end{pmatrix} \qquad N = 6\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{array}{l} \text{row} \\ \text{sums} \end{array} \begin{pmatrix} 12 \\ 6 \end{pmatrix}$$

(Waiting time for an '6's' is 12 steps - and for one '6' is 6 steps. c.f. p 114).

(d) To throw two '6's' and one '5':-

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ (0,0) & (1,0) & (0,1) & (1,1) & (0,2) & (1,2) \end{array} \\ \left(\begin{array}{ccccc:c} \frac{4}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{4}{6} & 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{4}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} \qquad \begin{array}{l} \text{state 6 is absorbing} \\ \\ \\ \\ \\ \\ \end{array}$$

We have $I - Q$

$$\frac{1}{6}\begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $N =$

$$\begin{pmatrix} 4 & 4 & 4 & 4 & 4 \\ & 1 & 0 & 1 & 0 \\ 0 & 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with row sums

$$6\begin{pmatrix} 24 \\ 2 \\ 14 \\ 1 \\ 1 \end{pmatrix} \text{ i.e. } \begin{pmatrix} 134 \\ 12 \\ 9 \\ 6 \\ 6 \end{pmatrix}$$

Waiting times are 13½, 12, 9, 6, 6 - the last four confirming earlier results. (c.f. p 116.)

(Some extensions are given in Appendix 2, p 165).

Example 2. (See p 27, the 'Christmas Tree Lights' problem)

We show on p 111 that, supposing that initially the set of 10 is complete, (and that prob. (failure) for a single bulb is $p = \frac{1}{36}$), the probability that K bulbs fail during a given year is given in the table.

K = 0	1	2	3 or more
$\left(\frac{35}{36}\right)^{10}$	$\binom{10}{1}\cdot\left(\frac{35}{36}\right)^{9}$	$45.\frac{1^2.35^8}{36^{10}}$	
0.7544	0.2136	0.0278	0.0022

Thus the transition matrix is P_{ij} where i is the number of failed bulbs at the beginning of the year, and j the number at the end.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \text{"3"} \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \text{"3"} \end{array} \\ \begin{pmatrix} 0.7545 & 0.2136 & 0.0277 & 0.0022 \\ 0 & 0.7545 & 0.2136 & 0.0299 \\ 0 & 0 & 0.7545 & 0.2455 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Then $I-Q = \begin{pmatrix} 0.2455 & -0.2136 & -0.0277 \\ 0 & 0.2455 & -0.2136 \\ 0 & 0 & 0.2455 \end{pmatrix}$ and N is $\begin{pmatrix} 4.073 & 3.577 & 3.600 \\ 0 & 4.073 & 3.577 \\ 0 & 0 & 4.073 \end{pmatrix}$

giving mean number of steps of 11.25, 7.65, and 4.07 respectively.

Thus, if no bulbs have failed, the set can be expected to last just over 11 years.

For the problem with 20 lights :-

The corresponding matrices are:-

$$Q = \begin{pmatrix} 0.5693 & 0.3090 & 0.0795 \\ 0 & 0.5693 & 0.3090 \\ 0 & 0 & 0.5693 \end{pmatrix} \qquad I-Q = \begin{pmatrix} 0.4307 & -0.3090 & -0.0795 \\ 0 & 0.4307 & -0.3090 \\ 0 & 0 & 0.4307 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 2.322 & 1.666 & 1.624 \\ 0 & 2.322 & 1.666 \\ 0 & 0 & 2.322 \end{pmatrix}$$

so that the expected number of years to failure is 5.611

Computer simulations (p. 144) based on three sets of 100 trials, gave corresponding figures of 10.59, 11.59, and 10.04 for 10 light bulbs and for 20 light bulbs (4 setsof 10 trials) 5.4, 5.6, 5.0, 5.1

Example 3. (see pp. 7, 107).

Collecting a set of 6 cards. Let i = state no. = no. of different cards collected.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{6} & \frac{4}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{6} & \frac{3}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{6} & \frac{2}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad I-Q = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & -\frac{5}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{6} & -\frac{4}{6} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{6} & -\frac{3}{6} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{6} & -\frac{2}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{pmatrix}$$

$$N = \begin{pmatrix} 1 & \frac{6}{5} & \frac{3}{2} & 2 & 3 & 6 \\ 0 & \frac{6}{5} & \frac{3}{2} & 2 & 3 & 6 \\ 0 & 0 & \frac{3}{2} & 2 & 3 & 6 \\ 0 & 0 & 0 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

Expected wait to collect set of 6 cards is $6 + 3 + 2 + \frac{3}{12} + \frac{6}{5} + 1 = 14.7$

(Compare p. 108 ; evidently the matrix solution would be very cumbersome for values of n>6).

Example 4. Rumour (see pp 1, 106).

We may consider a 7 state process, six in which 1,2,3,4,5 or 6 people know the rumour, and a seventh state in which the rumour ceases to be transmitted (i.e. state 7 is absorbing). The transition probability matrix is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{5} & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & \frac{3}{5} & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$I-Q = \frac{1}{5}\begin{pmatrix} 5 & -5 & 0 & 0 & 0 & 0 \\ 0 & 5 & -4 & 0 & 0 & 0 \\ 0 & 0 & 5 & -3 & 0 & 0 \\ 0 & 0 & 0 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

$$N = 5\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{125} & \frac{1}{625} & \frac{1}{3125} & \frac{1}{15625} \\ 0 & \frac{1}{5} & \frac{1}{25} & \frac{1}{125} & \frac{1}{625} & \frac{1}{3125} \\ 0 & 0 & \frac{1}{5} & \frac{1}{25} & \frac{1}{125} & \frac{1}{625} \\ 0 & 0 & 0 & \frac{1}{5} & \frac{1}{25} & \frac{1}{125} \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{25} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \end{pmatrix}$$

The row sums are

$\frac{2194}{625}$	i.e.	3.5104
$\frac{1569}{625}$		2.5104
$\frac{236}{125}$		1.888
$\frac{37}{25}$		1.48
$\frac{6}{5}$		1.2
1		1

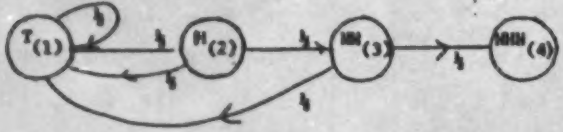
Thus mean no. of steps to absorption, starting in state 1, is 3.51

i.e. average no. of people who hear the rumour is 3.51.

(Compare p. 106 ; see also p. 146) Simulation by computer gave the average number hearing the rumour (based on 100 trials) of 3.41

Example 5. (c.f pp. 10, 108).

Toss a penny until a sequence of 3 consecutive heads, HHH, appears. Find the average number of tosses needed. The state diagram , transition matrix and associated matrices are as shown.

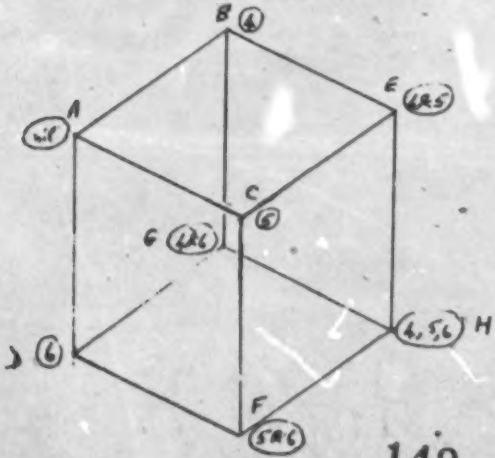


$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} ½ & ½ & 0 & 0 \\ ½ & 0 & ½ & 0 \\ ½ & 0 & 0 & ½ \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \qquad I-Q = \begin{pmatrix} ½ & -½ & 0 \\ -½ & 1 & -½ \\ -½ & 0 & 1 \end{pmatrix} \qquad N = \begin{pmatrix} 8 & 4 & 2 \\ 6 & 4 & 2 \\ 4 & 2 & 2 \end{pmatrix}$$

Waiting times from states 1,2,3 are 14,12,8 respectively.

Example 6. Waiting time to collect 4,5,6, in any order, on die (see p. 116)

The state diagram may be represented as a cube :-



The transition matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{array}{c} \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{6} & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = \frac{1}{6} \begin{pmatrix} 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

and $N = (I-Q)^{-1}$

$$= \begin{pmatrix} 2 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 3 & 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 3 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \quad \text{row sums} \quad \begin{pmatrix} 11 \\ 9 \\ 9 \\ 9 \\ 6 \\ 6 \\ 6 \end{pmatrix}$$

Hence the expected wait for (4,5,6), in any order, is 11

A computer simulation, (3 occasions, each of 20 experiments) produced the following results:-

Initial state	G	F	E	D	C	B	A
Expected wait	6	6	6	9	9	9	11
Observed wait	5.60	6.35	7.95	7.75	7.10	9.20	10.75
	7.35	4.65	5.00	9.45	8.35	9.95	12.00
	5.90	6.25	7.45	7.35	7.85	9.55	10.90

(average 11.216)

Example 7.

Snakes and ladders is played on a simplified board (see p. 90).

The transition matrix P is given by

States	1	2 (10)	3	7 (4)	5	6 (15)	8 (14)	9	12 (11)	16	13
1	0	½	½	0	0	0	0	0	0	0	0
2	0	0	½	½	0	0	0	0	0	0	0
3	0	0	0	½	½	0	0	0	0	0	0
4	0	0	0	0	0	0	½	½	0	0	0
5	0	0	0	½	0	½	0	0	0	0	0
6	0	0	0	½	0	0	½	0	0	0	0
(8)	0	0	0	0	0	½	0	0	0	½	0
9	0	½	0	0	0	0	0	0	½	0	0
12	0	0	0	0	0	0	½	0	0	0	½
16	0	0	0	0	0	0	0	0	0	1	0
13	0	0	0	0	0	0	0	0	0	0	1

P=

and the other matrices are calculated:-

$$N=(I-Q)^{-1} = \begin{pmatrix} 1 & .92 & .96 & 1.70 & .48 & 1.03 & 1.58 & .85 & .42 \\ 0 & 1.43 & .72 & 1.74 & .36 & .96 & 1.57 & .87 & .43 \\ 0 & .42 & 1.21 & 1.66 & .60 & 1.09 & 1.58 & .83 & .42 \\ 0 & .45 & .23 & 1.81 & .11 & .83 & 1.55 & .91 & .45 \\ 0 & .38 & .19 & 1.51 & 1.09 & 1.36 & 1.62 & .75 & .38 \\ 0 & .30 & .15 & 1.21 & .08 & 1.89 & 1.70 & .60 & .30 \\ 0 & .15 & .08 & .60 & .04 & .94 & 1.85 & .30 & .15 \\ 0 & .75 & .38 & 1.02 & .19 & .72 & 1.25 & 1.51 & .75 \\ 0 & .08 & .04 & .30 & .02 & .47 & .92 & .15 & 1.08 \end{pmatrix}$$

with row sums

$$\begin{pmatrix} 8.94 \\ 8.08 \\ 7.81 \\ 6.34 \\ 7.28 \\ 6.23 \\ 4.11 \\ 6.57 \\ 7.06 \end{pmatrix}$$

Thus the average length of a game is 8.9 throws.

Also R is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{3} & 0 \\ 0 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ and NR is $\begin{pmatrix} .79 & .21 \\ .78 & .22 \\ .79 & .21 \\ .77 & .23 \\ .81 & .19 \\ .85 & .15 \\ .92 & .08 \\ .62 & .38 \\ .46 & .54 \end{pmatrix}$

giving the probability of a win as 0.7[illegible].

(The probability of a win, given that one is on square '8', is 0.92, and from square '12' is 0.46).

The results of computer simulations of the game were as follows (for 100 experiments)

Mean numbers of times in each state, starting in state 1, was

(1, .88, .88, 1.65, .44, 1.02, 1.63, .71, .40)

and mean number of steps to absorption, from each possible starting state, was'

(8.61, 7.14, 8.15, 6.71, 7.38, 6.70, 3.35, 4.48, 2.79)

Frequencies of absorption were:-

from state	1	2	3	4	5	6	8	9	12
to '16'	83	73	84	71	84	84	95	56	44
to '13'	17	27	16	29	16	16	5	44	56

Example 8. Pinball Machines (see p 26).

The matrices for the simplified example (a), which has one absorbing state C, are

$$P = \begin{array}{c} \\ A \\ B \\ D \\ C \end{array} \begin{array}{c} \begin{array}{cccc} A & B & D & C \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & 0 & 0 & \frac{5}{6} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad I-Q = \frac{1}{6}\begin{pmatrix} 6 & -3 & 0 \\ 0 & 6 & -3 \\ -1 & 0 & 6 \end{pmatrix} \quad N = \frac{1}{23}\begin{pmatrix} 24 & 12 & 6 \\ 2 & 24 & 12 \\ 4 & 2 & 24 \end{pmatrix}$$

Mean no. of steps to absorption = $\frac{42}{23} = 1.826$

For example (b) in which E,F,G are the absorbing states.

$$P = \begin{array}{c|cccc|ccc} & A & B & C & D & E & F & G \\ A & 0 & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ C & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ D & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{5}{6} \\ \hline E & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ F & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ G & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$I-Q = \frac{1}{6}\begin{pmatrix} 6 & -3 & -3 & 0 \\ 0 & 6 & 0 & -3 \\ 0 & 0 & 6 & 0 \\ -1 & 0 & 0 & 6 \end{pmatrix} \qquad R = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{5}{6} \end{pmatrix}$$

$$N = (I-Q)^{-1} = \frac{1}{23}\begin{pmatrix} 24 & 12 & 12 & 6 \\ 2 & 24 & 1 & 12 \\ 0 & 0 & 23 & 0 \\ 4 & 2 & 2 & 4 \end{pmatrix} \qquad NR = \frac{1}{69}\begin{pmatrix} 30 & 24 & 15 \\ 37 & 2 & 30 \\ 23 & 46 & 0 \\ 5 & 4 & 60 \end{pmatrix}$$

Mean no. of steps to absorption is $\frac{54}{23} = 2.348$

and the absorption probabilities are as shown:-

$$\begin{pmatrix} .43 & .35 & .22 \\ .54 & .03 & .43 \\ .33 & .67 & 0 \\ .07 & .06 & .87 \end{pmatrix}$$

Expected score for game $= \frac{12}{23}.10 + \frac{12}{23}.30 + \frac{6}{23}.10 + \frac{30}{69}.20 + \frac{24}{69}.10 + \frac{15}{69}.0$

(B) (C) (D) — transient states; (E) (F) (G) — absorbing states

$$= \frac{120 + 360 + 60 + 200 + 80}{23} = 35.65$$

Computer simulation (2 sets of 100 trials) produced the following outcomes:-
(Sim. 144, 156)

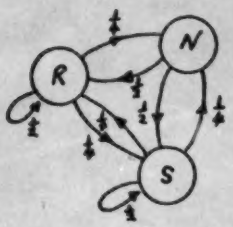
	(i)				(ii)			
Mean number of times in each state	1.02	0.46	0.36	0.22	1.03	0.42	0.61	0.20
	0.08	1.04	0.04	0.49	0.09	1.05	0.04	0.53
	0	0	1	0	0	0	1	0
	0.16	0.07	0.09	1.04	0.19	0.13	0.06	1.07
Mean number of steps to absorption (from A)	2.26				2.26			
Number of times absorbed (from A)	E35, F45, G20				E40, F43, G17			

Example 9

Weather in the land of Oz. (from Kemeny et al, 1958).

"In the land of Oz they never have two nice days in a row. After a nice day they are just as likely to have snow as rain next day. After snow (or rain) there is an even chance of the same weather next day. If there is a change from snow or rain, only half of the time is there a change to a nice day".

The probabilities of moving from state to state, shown in the diagram, may be expressed in matrix form, P.



$$P = \begin{array}{c} \\ \text{1st state} \end{array} \begin{array}{c} \begin{array}{ccc} \text{1r} & \text{2n} & \text{3s} \end{array} \\ \begin{pmatrix} ½ & ¼ & ¼ \\ ½ & 0 & ½ \\ ¼ & ¼ & ½ \end{pmatrix} \end{array} \quad \text{2nd state}$$

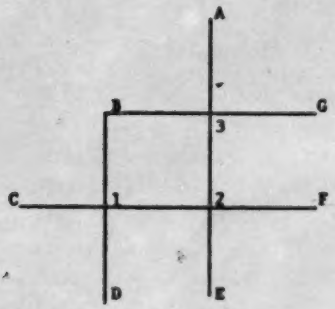
There are no absorbing states. By considering the value of P for large n, we can estimate the proportion of time spent in each state, namely (0.4, 0.2, 0.4).

$$P^n \longrightarrow \begin{pmatrix} .4 & .2 & .4 \\ .4 & .2 & .4 \\ .4 & .2 & .4 \end{pmatrix} \quad \text{as } n \to \infty$$

Example 10

Random walk on a grid (from Kemeny et al, 1958).

Suppose a particle starts from the point '1' and moves one step, having equal probability of going N,S,E,W. At each of the points 1,2,3, it has four equally probable choices for the next step. If, however, it reaches one of the points A,B,C,D,E,F,G, it remains there, ("is absorbed").



The matrix of transition probabilities is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & A & B & C & D & E & F & G \end{array} \\ \begin{pmatrix} 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$Q \text{ is } \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$$

$$\text{and } R \text{ is } \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$N \text{ is } \frac{1}{14}\begin{pmatrix} 15 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix}$$

Hence average number of steps to absorption, is $\frac{20}{14} = 1.43$ from state 1

$\frac{24}{14} = 1.71$ from state 2

$\frac{20}{14} = 1.43$ from state 3

$$NR = \frac{1}{56}\begin{array}{c} \begin{matrix} A & B & C & D & E & F & G \end{matrix} \\ \begin{pmatrix} 1 & 16 & 15 & 15 & 4 & 4 & 1 \\ 4 & 8 & 4 & 4 & 16 & 16 & 4 \\ 15 & 16 & 1 & 1 & 4 & 4 & 15 \end{pmatrix} \end{array}$$

Thus e.g. Probability of absorption at A, starting in state 1, is $\frac{1}{56}$

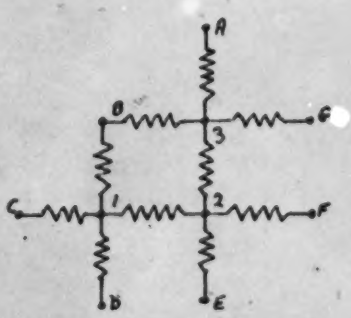
and probability of absorption at B, starting in state 1, is $\frac{16}{56}$ etc.

(Kemeny et al point out that the circuit illustrated below enables these probabilities to be found by electrical measurements - for the voltage at each interior point, '1', '2', '3' is proportional to the probability of being absorbed at A in the random walk, starting at the corresponding interior point. This is so because the set of equations, governing the interior voltages, involves the same matrices N and R. Thus the circuit acts as an analogue computer).

Voltage at A is 1 *volt*.

Voltages at B,C,D,E,F,G are all zero.

All resistances of equal value.



Producing powers of matrices

It was shown earlier (p. 125) that in a Markov process, the probability of being in state j, N steps after starting in state i, is given by the (i,j)th entry of the matrix P^N (where P is the matrix of transition probabilities).

In order to calculate powers of matrices, the BASIC program MATPOWER, given below, may be used. The powers of P which are output, P^I, may be either such that values of I are in the geometric sequence 1, 2, 4, 8, 16, 32, . . . or in an arithmetic sequence e.g. 3, 6, 9, 12, 15, . . . The first is useful in examining the limiting behaviour of P^N for large N (P^{32} is easily calculated as $P^{16} \times P^{16}$), the second when the whole range of powers is of interest. The 'step' M, in the arithmetic sequence, may be altered at will (in the case above, the common difference, M, is 3). (The program first calculates and prints out $P^2, P^3, \ldots P^M$, and then $P^{2M}, P^{3M}, \ldots P^{KM}$, where both M and K are input during the run).

Some sample runs are given after the listing, producing output relevant to problems considered earlier in this section. (The program MATPWRT which precedes MATPOWER is a simpler version, which may be helpful in understanding the operation of MATPOWER itself). Data input for many of the problems in this section is amalgamated in the file DATMAT, (listed on p. 149). If the MERGE and RESEQUENCE facilities (p. 101) are available, this may be used as follows:

Example
read in DATMAT
delete lines 1-190, 300-1500
RESEQUENCE to change lines from 200 - 270 to 500 - 570
MERGE with MATPOWER

(In this case it is necessary to add the instruction MAT X = (1/6) * X, say as line 55, since $P = \frac{1}{6}\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$, and to change line 59 to MAT PRINT X(L,L).)

LIST to check that all is well
RUN

giving powers of the transition matrix on p 123. (The input value L is 6).

The program MATDN is a minor adaptation of MATPOWER which may be used to study absorption probabilities. Thus if L is an absorbing state, the (1,L)th element of P^N gives the probability that the process, starting in state 1, has moved to state L by the time N steps have been taken - from which we can determine, for example, the cumulative probability distribution of game lengths in the dice-game problem (p. 123), of waiting times in collecting '4', '5' and '6' (p. 137), or of life expectancy in the Christmas tree lights problem (p. 134).

These are given in the sample runs of MATDN which follow the listing and may be compared with the simulation results for the same problems on pp. 129, 138 and 135. (To avoid having to type in data for the matrix P, the process of MERGEing with the file DATMAT, as described above, may be used)

A program for Markov chains

The program MARKOV listed on pp. 147-9 is designed to calculate the matrices giving theoretical results — i.e. Q, N = $(I - Q)^{-1}$, and NR — for a given Markov chain process, whose transition matrix P is supplied as data, and also the results of a simulation, repeated E times (in the listing, E = 100), in which each non-absorbing state is selected, in turn, as the starting state, and the process is followed until an absorbing state is reached. The "observed" matrices corresponding to N (mean number of times in each state), row sums of N (mean no. of steps to absorption) and NR (probabilities of absorption in each state) are printed out for comparison with the theoretical values. The listing and sample runs (referring to the problems P... 4.11, Diagram 2 on pp. 140, 137) are given below. MARKOV is a "framework" program - data may be added in lines 180 - 200, and it may also be convenient to insert a line MAT X= (1/6) * X or MAT X= (1/2) * X to simplify typing in the input data. (The method of MERGEing with a portion of the file DATMAT is again useful).

MATPRT

(Input a matrix and compute its first 7 powers)

```
0010 PRINT "INPUT DIM OF MATRIX"
0020 INPUT L
0030 PRINT L
0040 PRINT "INPUT MATRIX"
0050 MATINPUT X(L,L)
0060 MATPRINT X;          To display input
0180
0190
0191 MAT P=X*X            P = X^2
0192 MAT Q=X*P            Q = X^3
0193 MAT R=X*Q            R = X^4
0194 MATPRINT USING "10.3#", P;Q;R;
0195 MAT S=X*R            S = X^5
0196 MAT T=X*S            T = X^6
0197 MAT U=X*T            U = X^7
0198 MATPRINT USING "10.3#", S;T;U;
0200
0210
0220 END
```

Example of a run:

```
MATPRT  04 JUN 1980 12:47:01
INPUT DIM OF MATRIX
?
 2
INPUT MATRIX
?

 0.1000000  0.2000000          X
 0.3000000  0.4000000

 0.070  0.100          X^2
 0.150  0.220

 0.037  0.054          X^3
 0.081  0.118

 0.020  0.029          X^4
 0.043  0.063

 0.011  0.016          X^5
 0.023  0.034

 0.006  0.008          X^6
 0.013  0.018

 0.003  0.004          X^7
 0.007  0.010

READY
```

MATPOWER

Output either $X^n, X^{2n}, X^{3n}, \ldots (AP.)$ or $X, X^1, X^2, X^3, \ldots (G.P.)$

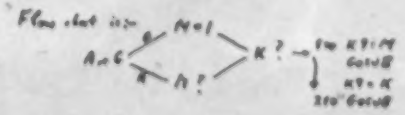
```
0010 PRINT "INPUT DIM OF MATRIX"
0020 INPUT L
0030 PRINT L
0040 PRINT "INPUT MATRIX"
0050 MATINPUT X(L,L)          (could be modified to "Read")
0060 MATPRINT X;
0070 PRINT "POWERS IN ARITHMETIC OR GEOMETRIC PROGRESSION?"
0080 PRINT "TYPE  A    OR    G "
0090 INPUT A$
0100 IF A$="G" THEN N=1:GOTO 140
0110 PRINT "TYPE N FOR EVERY NTH TERM OF SEQUENCE"
0120 INPUT N
0130 PRINT N
0140 PRINT "HOW MANY POWERS WANTED?"
0150 INPUT K
0160 PRINT K
0170 K9=N
0180 GOSUB 1000 ------------> Calculate X^N
0190 MAT X=Z
0200 K9=K
0210 GOSUB 1000 ------------> Calculate either X^N, X^2N, X^3N, ...  AP if A$ ≠ "G"
                                  or X, X^2, X^3, ...  GP if A$ = "G"
                                  (= N=1)
0220 END

1000 MAT T=X   ) copy X
1010 MAT T=X
1020 FOR J=1 TO K9
1030 IF A$="G" THEN MAT T=Y     (in this case give Y^2 rather than Y.X)
1040 MAT Z=Y*T   Z.Y^2 or Z.Y.X
1050 MAT Y=Z     update Y
1060 MATPRINT USING "20.38",Z;
1070 NEXT J
1080 RETURN
```

Notes.

Both X, Y, Z, T must be given (default value used)

Flow chart is:



Variables

A$ = 'A' or 'G' for AP/G.P.

X = input matrix, square, dimension L

N = gap between number of powers. (2 G.P. N is not 1)

K = no. of powers printed out – so for X^{Nn} – A.P.
(2^n) – G.P.

K9 control pint for T = GOSUB
Set at N in 1st call – to generate X, X^2, X^3 (K-3)
Then set at K to generate – $X^N, X^{2N}, X^{3N}, \ldots$
$X, X^2, X^3, X^4, \ldots$

Z = current product

Y = last value of current product

T = X = AP.
Y = G.P.
] this 1080.

MATRIX Exhibits the $(1,L)^{th}$ element of successive powers of the Markov matrix, where the L^{th} state is absorbing. Adapted from method and notation given in reference 9, p. 880.

```
0010 PRINT "INPUT DIM OF MATRIX"
0020 INPUT L
0030 PRINT L
0040 PRINT "INPUT MATRIX"
0050 MATREAD X(L,L)
0060 REM INSERT HERE EG.MAT X =(1/6)*X
0070 MATPRINT X;     (as check)
0080 PRINT "TYPE N FOR EVERY NTH TERM OF SEQUENCE"
0090 INPUT N
0100 PRINT N
0110 PRINT "HOW MANY POWERS WANTED?"
0120 INPUT K
0130 PRINT K
0140 PRINT "TERM";"          ";"PROBY.";"          ";"CUM DISTN."
0150 DIM A(100)
0160 MAT A=ZER(K)
0170 IF N=1 THEN A(0)=X(1,L):GOTO 320
0180 K9=K
0190 GOSUB 370
0200 MAT B=Z
0210 A(0)=X(1,L)
0220 K9=K
0230 PRINT N;"          ";
0240 PRINT "          ";
0250 PRINT USING "10.3#",A(0),
0260 PRINT "          ";
0270 PRINT USING "10.3#",A(0)
0280 GOSUB 370
0290 FOR J=1 TO K9-1
0300 PRINT N*(J+1),
0310 PRINT "          ";
0320 PRINT USING "10.3#",A(J)-A(J-1),
0330 PRINT "          ";
0340 PRINT USING "10.3#",A(J)
0350 NEXT J
0360 END
0370 MAT Y=X
0380 MAT T=X
0390 FOR J=1 TO K9-1
0400 MAT B=Y*T
0410 MAT Y=B
0420 A(J)=B(1,L)
0430 NEXT J
0440 RETURN
```

initialise product X_0 matrix

this bit

successive powers of X^N

incremental probability

cumulative probability

multiply

Matrix X is multiplied by itself successively using nested for-next loops, which are temporarily adjusted (p.60).

Here is where to insert N for every Nth term of sequence — as in Markov chains (p.170).

Variables

L dimension of Markovian matrix X, (L^{th} state is absorbing)

M,K checks of $X^2, X^3, \ldots, X^{K9}$ as printed out

K9 control quantity of matrix at 370 (Print for J)

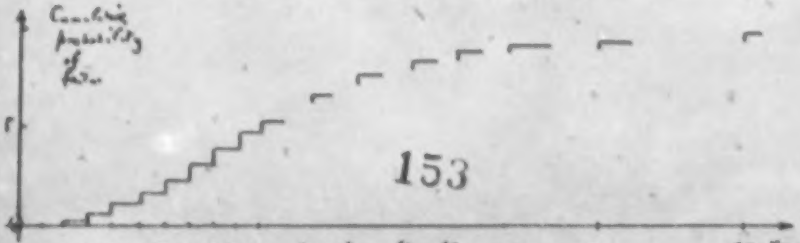
Z unit matrix

Y Electronic of second product

T copy of X

A(J) = the probability of being absorbed in state L, from starting state 1, after J steps

MATRIX applied to the Christmas Eggs problem, p.186.



$P(X_3)$ = 0.07 0.05 0.08 0.08 0.09 0.11 0.14 0.18 0.24 0.31 0.39 0.48 0.58 0.68 0.78

Now easily way to expand till matrix can obtain experimentally, by recalling - open RANDOM [79]

DATAMAT - cf. ch. 6. (suppl)

```
0100 REM XMAS TREE 10BULBS p134
0120 DATA .7544933,.2155695,.0277161,.0022211
0130 DATA 0,.7544933,.2155695,.0299372
0140DATA 0,0,.7544933,.2455067,
0150 DATA 0,0,0,1
0200 REM GAME OF p123 TWO 1'S AND ONE ODD
0210
0220 DATA 2,1,2,0,0,0
0230 DATA 0,2,0,1,2,0
0240 DATA 0,0,2,0,1,0
0250 DATA 0,0,0,2,0,2
0260 DATA  0,0,0,0,2,1
0270 DATA 0,0,0,0,0,2
0300 REM XMAS TREE 20 LIGHTS ,p134
0310
0320 DATA .5693,.3090,.0795,.0422
0330 DATA 0,.5693,.3090,.1217
0340 DATA 0,0,.5693,.4387
0350 DATA 0,0,0,1
0400 REM WAIT 4,5,6 PROBLEM p137
0410
0420 DATA 5,0,0,0,0,0,0,1
0430 DATA 0,5,0,0 0,0,0,1
0440 DATA 0,0,5,0 0,0,0,1
0450 DATA 1,1,0,4,0,0,0,0
0460 DATA 0,1,1,0,4,0,0,0
0470 DATA 1,0,1,0,0,4,0 ,0
0480 DATA 0,0,0,1,1,1,3,0
0490 DATA 0,0,0,0,0,0,0,6
0500 REM  Dice game,p129
0510
0520 DATA 3,1,0,0,2,0
0530 DATA 0,3,1,2,0,0
0540 DATA 0,0,4,0,0,2
0550 DATA 0,0,0,5,0,1
0560 DATA 0,0,0,0,6,0
0570 DATA 0,0,0,0,0,6
0600 REM XMAS TREE 10 BULBS P=0.03(see p 134)
0610
0620 DATA .7374241,.2280693,.0317416,.002765
0630 DATA 0,.7374241,.2280693,.0345066
0640 DATA 0,0,.7374241,.2625759
0650 DATA 0,0,0,1
0700 REM 8 AND LADDERS p139
0710
0720 DATA 0,1,1,0,0,0,0,0,0,0,0,0
0730 DATA 0,0,1,1,0,0,0,0,0,0,0,0
0740 DATA 0,0,0,1,1,0,0,0,0,0,0,0
0750 DATA 0,0,0,0,0,0,1,1,0,0,0
0760 DATA 0,0,0,1,0,1,0,0,0,0,0,0
0770 DATA 0,0,0,1,0,0,1,0,0,0,0,0
0780 DATA 0,0,0,0,0,1,0,0,0,1,0
0790 DATA 0,1,0,0,0,0,0,0,1,0,0
0800 DATA 0,0,0,0,0,0,0,1,0,0,0,1
0810 DATA 0,0,0,0,0,0,0,0,0,0,2,0
0820 DATA 0,0,0,0,0,0,0,0,0,0,0,2
```

```
0900 REM GUILE NB P=1/6*P Pinball game.p140
0910 DATA 0,3,3,0,0,0,0
0920 DATA 0,0,0,3,3,0,0
0930 DATA 0,0,0,0,0,2,4,0
0940 DATA 1,0,0,0,0,0,0,5
0950 DATA 0,0,0,0,0,6,0,0
0960 DATA 0,0,0,0,0,0,6,0
0970 DATA 0,0,0,0,0,0,0,6
0990 REM DRUNK  c.f. p55
1000 DATA 0,1,0,0,0,0,0,0,1
1010 DATA 1,0,1,0,0,0,0,0,0
1020 DATA  0,1,0,1,0,0,0,0,0
1030 DATA  0,0,1,0,1,0,0,0,0
1040 DATA 0,0,0,1,0,1,0,0,0
1050 DATA 0,0,0,0,1,0,1,0,0
1060 DATA 0,0,0,0,0,1,0,1,0
1070 DATA 0,0,0,0,0,0,0,2,0
1080 DATA 0,0,0,0,0,0,0,0,2
1100 REM RUMOUR p136
1110 DATA 0,4,0,0,0,1
1120 DATA 0,0,3,0,0,2
1130 DATA 0,0,0,2,0,3
1140 DATA 0,0,0,0,1,4
1150 DATA 0,0,0,0,0,5
1160 DATA 0,0,0,0,0,5
```

```
1200 REM OCTOPUS Appendix II
1220 DATA 4,2,0 ,0,0,0,0,0,0,0,0,0,0,0,
1230 DATA 0,4,1,1,0,0,0,0,0,0,0,0,0,0,0
1240 DATA 0,0,4,0,1,1,0,0,0,0,0,0,0,0,0
1250 DATA 0,0,0,4,0,2,0,0,0,0,0,0,0,0,0
1260 DATA 0,0,0,0,4,0,1,1,0,0,0,0,0,0,0
1270 DATA 0,0,0,0,0,4,0,1,1,0,0,0,0,0,0
1280 DATA 0,0,0,0,0,0,5,0,0,1,0,0,0,0,0
1290 DATA 0,0,0,0,0,0,0,4,0,1,1,0,0,0,0
1300 DATA 0,0,0,0,0,0,0,0,4,0,2,0,0,0,0
1310 DATA 0,0,0,0,0,0,0,0,0,5,0,1,0,0,0
1320 DATA 0,0,0,0,0,0,0,0,0,0,4,1,1,0,0
1330 DATA 0,0,0,0,0,0,0,0,0,0,5,0,1,0
1340 DATA 0,0,0,0,0,0,0,0,0,0,0,4,2,0
1350 DATA 0,0,0,0,0,0,0,0,0,0,0,0,5,1
1360 DATA 0,0,0,0,0,0,0,0,0,0,0,0,0,6
```

PURPOSE: Calculate theoretical probabilities associated with a Markov chain process (p 400-700) and perform simulation using same non-absorbing state as starting state.

Reference: Kemeny et al (1958) p 64ff

```
0010 REM MARKOV SIMULATION
0020 DIM P(20,20)
0030 DIM C(20,1)
0040READ A,L,E
0050 T=L-A
0060 A1$="20.20"      ('picture', for PRINT USING - see P. 370)
0070 MAT READ P(L,L)
0080REM DATA VALUES OF A,L,E IN LINE 100
0090 REM DATA FOR P FOLLOWS INLINES 110-300
0100
0110
0120
0130
0140
0150
0160
0170
0180
0190
0200
0300
0310
0320 MAT U=ZER(T,T)
0330 MAT V=ZER(T)
0340 MAT W=ZER(T,A)
0350 REM MIGHT INSERT HERE MAT P=(1/6)*P
0360 PRINT "PROB MATRIX"
0370 MAT PRINT USING A1$, P;
0380
0390
0400
0410 MAT Q=ZER(T,T)
0420 MAT R=ZER(T,A)
0430MAT N=ZER(T,T)
0440
0450
0460 REM INSERT GOTO 810
0470 REM ON NEXT LINE TO OMIT THEORETICAL CALCULATIONS
0480 FOR I=1 TO T
0490 FOR J=1 TO T
0500 Q(I,J)=P(I,J)
0510 NEXT J
0520
0530 FOR J=1 TO A
0540 R(I,J)=P(I,J+T)
0550 NEXT J,I
0560 MATPRINT USING A1$, Q;
0570 MATPRINT USING A1$, R;
0580
0590 MAT N=IDN(T,T)
0600 MATREAD C(T,1)
0610
0620
0630
0640
0650 MAT N=N-Q
0660 MAT N=INV(N)
0670 PRINT "FUNDL MATRIX"
0680 MAT PRINT USING A1$, N;
```

Variables

E = no. of simulations (if 'picture' format)
A = no. of absorbing states
L = total no of states
T = no. of transient states
P = transition matrix = $\begin{pmatrix} Q & R \\ O & I \end{pmatrix}$
Q is T by T (Identity)
R is T by A (Q zero)
N = $(I-Q)^{-1}$ = fundamental matrix.
(gives no. of times in each state)
NoC = row sums of N → expected time to absorption
NoR = sums of absorbing cols of Q → probs of absorption in each A state.
U, V, W = work matrix comp. N, NoC, NoR
C = matrix, T by 1, all 1's ($\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$)

(Further data can be input in (row) vector form)

Put P in Q and R

Q and R

Display Q, R

Matrix used to calculate I - Q.

Reserve write, Next from lines 1170, 1180 at end.

Now N = I - Q

and N = $(I-Q)^{-1}$

(contd) →

(Markov)

```
 0690
 0700 PRINT "ABSORPTION TIMES"
 0710
 0720 MAT X=N*C
 0730
 0740 MATPRINT USING A1$, X;
 0750 MAT Y =N*R
 0760 PRINT "ABSORPTION PROBS"
 0770 MAT PRINT USING A1$, Y;
 0780
 0790
 0800
 0810 REM BEGINNING OF SIMULATION
 0820 RANDOMISE
 0830
 0840 PRINT "RESULTS OF ";E;" SIMULATIONS"
 0850 REM PERFORM E EXPTS  FOR EACH OF NON ABSORBING STATES
 0860 FOR S1=1 TO T
 0870 FOR C1 =1 TO E
 0880 D=0          counter
 0890 S=S1
 0900 REM S=CURRENT STATE  :  SELECT NEW STATE
 0910 R1=RND
 0920
 0930 N=0
 0940 FOR I=1TO L
 0950 N=N+P(S,I)
 0960 IF R1<N THEN S=I:GOTO 1000
 0970 NEXT I
 0980
 0990 REM NEW STATE
 1000 D=D+1          no. of moves made so far
 1010 REM NO OF MOVES MADE IN CURRENT EXPT
 1020 IF S>T THEN 1070     jump if absorbed
 1030 REM NOT ABSORBED.  RECORD NEW STATE
 1040 U(S1,S)=U(S1,S)+1    augment count in new state
 1050 GOTO 910
 1060 REM ABSORBED
 1070 V(S1)=V(S1)+D     absorbed in D steps, from starting state S1
 1080 REM RECORD NO OF STEPS TO CAPTURE AND ABSORBING STATE
 1090
 1100 W(S1,S-T)=W(S1,S-T)+1     absorbed in S, the starting in state S1.
 1110 NEXT C1
 1120 NEXT S1
 1130
 1140
 1150 REM OUTPUT RESULTS OF EXPTS
 1160 K=1/E              & average over E expts.
 1170 MAT U=(K)*U
 1180 PRINT "MEAN NO OF TIMES EACH STATE"
 1190 MAT PRINT USING A1$, U;
 1200 MAT V=(K)*V
 1210 PRINT "MEAN NO OF STEPS TO ABSORPTION"
 1220 MAT PRINT USING A1$, V;
 1230
 1240 MAT W = (K)*W
 1250 PRINT "PROBS OF ABSORPTION IN GIVEN STATES"
 1260 MAT PRINT USING A1$, W;
 1270 DATA 1,1,1,1,1,1,1,1,1,1
 1280 DATA 1,1,1,1,1,1,1,1,1,1,1,1,1
 1290 END
```

Variables (contd.)

S1 } control variables { state S1-T, then exit
C1 }

S = current state

T = total number representing the absorbing states

L = number of exits - one of pairs for each exit P

U(A,B) = no. of times state B is entered if initial point in was state A.

V(A) = Σ no. of steps to absorption starting in A

W(A,B) = no. of times absorbed in B

R1 = random no. (a starting value will R)

P = Σ of probabilities across row of matrix, = double transition

* Note - diagonal values of P should also be accessed by I & give "mean no. of times in each state" (l. 1180)

[e.g. Mean(T,T) = P.220]

to find

156

MARKOV simulation of PINBALL problem (p141)
OUTPUT: Theoretical

FROM MATRIX

```
0.00 0.50 0.50 0.00 0.00 0.00 0.00
0.00 0.00 0.00 0.50 0.50 0.00 0.00
0.00 0.00 0.00 0.00 0.33 0.67 0.00
0.17 0.00 0.00 0.00 0.00 0.00 0.83
0.00 0.00 0.00 0.00 1.00 0.00 0.00
0.00 0.00 0.00 0.00 0.00 1.00 0.00
0.00 0.00 0.00 0.00 0.00 0.00 1.00
```

P

```
0.00 0.50 0.50 0.00
0.00 0.00 0.00 0.50
0.00 0.00 0.00 0.00
0.17 0.00 0.00 0.00
```

Q

```
0.00 0.00 0.00
0.50 0.00 0.00
0.33 0.67 0.00
0.00 0.00 0.83
```

R

FUNDL MATRIX

```
1.04 0.52 0.52 0.26
0.09 1.04 0.04 0.52
0.00 0.00 1.00 0.00
0.17 0.09 0.09 1.04
```

N

ABSORPTION TIMES

```
2.35
1.78
1.00
1.39
```

N·C

ABSORPTION PROBS

```
0.43 0.35 0.22
0.54 0.03 0.43
0.33 0.67 0.00
0.07 0.04 0.87
```

N·R

RESULTS OF 100 SIMULATIONS ← "experimental"
MEAN NO OF TIMES EACH STATE

```
0.00 0.50 0.50 0.24
0.00 0.04 0.04 0.58
0.00 0.00 0.00 0.00
0.19 0.09 0.10 0.03
```

compare with N

MEAN NO OF STEPS TO ABSORPTION

```
2.24 1.74 1.00 1.41
```

cf. N·C

PROBS OF ABSORPTION IN GIVEN STATES

```
0.38 0.38 0.24
0.48 0.02 0.50
0.33 0.67 0.00
0.11 0.05 0.84
```

cf. N·R

← second batch

RESULTS OF 100 SIMULATIONS
MEAN NO OF TIMES EACH STATE

```
0.05 0.51 0.54 0.24
0.10 0.06 0.04 0.53
0.00 0.00 0.00 0.00
0.17 0.07 0.10 0.04
```

MEAN NO OF STEPS TO ABSORPTION

```
2.34 1.71 1.00 1.38
```

PROBS OF ABSORPTION IN GIVEN STATES

```
0.43 0.38 0.19
0.54 0.03 0.43
0.29 0.71 0.00
0.06 0.07 0.87
```

MARKOV output for DGAME2 problem (p131)

PROB MATRIX

```
0.50 0.17 0.00 0.00 0.33 0.00
0.00 0.50 0.17 0.33 0.00 0.00
0.00 0.00 0.67 0.00 0.00 0.33
0.00 0.00 0.00 0.83 0.00 0.17
0.00 0.00 0.00 0.00 1.00 0.00
0.00 0.00 0.00 0.00 0.00 1.00
```

```
0.50 0.17 0.00 0.00
0.00 0.50 0.17 0.33
0.00 0.00 0.67 0.00
0.00 0.00 0.00 0.83
```

```
0.33 0.00
0.00 0.00
0.00 0.33
0.00 0.17
```

FUNDL MATRIX

```
2.00 0.67 0.33 1.33
0.00 2.00 1.00 4.00
0.00 0.00 3.00 0.00
0.00 0.00 0.00 6.00
```

ABSORPTION TIMES

```
4.33
7.00
3.00
6.00
```

ABSORPTION PROBS

```
0.67 0.33
0.00 1.00
0.00 1.00
0.00 1.00
```

RESULTS OF 100 SIMULATIONS
MEAN NO OF TIMES EACH STATE

```
1.15 0.82 0.40 1.28
0.00 0.89 0.82 4.70
0.00 0.00 1.74 0.00
0.00 0.00 0.00 4.47
```

MEAN NO OF STEPS TO ABSORPTION

```
4.65 7.41 2.74 5.47
```

PROBS OF ABSORPTION IN GIVEN STATES

```
0.61 0.39
0.00 1.00
0.00 1.00
0.00 1.00
```

another run :

RESULTS OF 100 SIMULATIONS
MEAN NO OF TIMES EACH STATE

```
0.85 0.69 0.49 0.95
0.00 0.95 1.02 4.29
0.00 0.00 2.59 0.00
0.00 0.00 0.00 5.27
```

MEAN NO OF STEPS TO ABSORPTION

```
3.98 7.26 3.59 6.27
```

PROBS OF ABSORPTION IN GIVEN STATES

```
0.65 0.35
0.00 1.00
0.00 1.00
0.00 1.00
```

Examples of use of 'MAT04'

(a) A Comp - A 123

```
MAT200   04 JUN 1980 12:23:28
INPUT DIM OF MATRIX
?
 6
INPUT MATRIX

 0.4999999  0.1666666  0.3333333  0  0  0
 0  0.4999999  0  0.1666666  0.3333333  0
 0  0  0.8333331  0  0.1666666  0
 0  0  0  0.6666665  0  0.3333332
 0  0  0  0  0.8333331  0.1666666
 0  0  0  0  0  0.9999990

TYPE N FOR EVERY NTH TERM OF SEQUENCE
?
 2
HOW MANY POWERS WANTED?
?
 20
TERM                    PROBY.          CUM DISTN.
 2                      0.000           0.000
 4                      0.080           0.080
 6                      0.142           0.222
 8                      0.148           0.371
 10                     0.132           0.502
 12                     0.110           0.612
 14                     0.089           0.701
 16                     0.070           0.771
 18                     0.055           0.827
 20                     0.043           0.869
 22                     0.033           0.902
 24                     0.025           0.927
 26                     0.019           0.946
 28                     0.014           0.960
 30                     0.011           0.970
 32                     0.008           0.978
 34                     0.006           0.984
 36                     0.004           0.988
 38                     0.003           0.992
 40                     0.002           0.994
READY
TIME USED:     0 MINS
'BASIC' SUBSYSTEM LEFT AT: 04 JUN 1980 12:23:37
```

(b) Christmas Tree 10 6-Pts A736

Term	Proby	Cum Distn
3	0.050	0.050
6	0.183	0.233
9	0.225	0.458
12	0.193	0.651
15	0.138	0.789
18	0.089	0.879
21	0.054	0.933
24	0.031	0.964
27	0.017	0.981
30	0.009	0.990

Use of MATRIX
(2) 4,5,6 Prob Chap. 6 197

```
READY
RUN2400   04 JUN 1980 12:33:12
INPUT DIM OF MATRIX
?
 8
INPUT MATRIX

 0.8333331  0  0  0  0  0  0  0.1666666
 0  0.8333331  0  0  0  0  0  0.1666666
 0  0  0.8333331  0  0  0  0  0.1666666
 0.1666666  0.1666666  0  0.6666665  0  0  0  0
 0  0.1666666  0.1666666  0  0.6666665  0  0  0
 0.1666666  0  0.1666666  0  0  0.6666665  0  0
 0  0  0  0.1666666  0.1666666  0.1666666  0.4999999  0
 0  0  0  0  0  0  0  0.9999998


TYPE N FOR EVERY NTH TERM OF SEQUENCE
?
 1
HOW MANY POWERS WANTED?
?
 25
```

TERM	PROBY.	CUM DISTN.
1	0.167	0.167
2	0.139	0.306
3	0.116	0.421
4	0.096	0.518
5	0.080	0.598
6	0.067	0.665
7	0.056	0.721
8	0.047	0.767
9	0.039	0.806
10	0.032	0.838
11	0.027	0.865
12	0.022	0.888
13	0.019	0.907
14	0.016	0.922
15	0.013	0.935
16	0.011	0.946
17	0.009	0.955
18	0.008	0.962
19	0.006	0.969
20	0.005	0.974
21	0.004	0.978
22	0.004	0.982
23	0.003	0.985
24	0.003	0.987
25	0.002	0.990

```
READY
TIME USED:    0 MINS
'BASIC' SUBSYSTEM LEFT AT: 04 JUN 1980 12:33:24
```

Output from MATPOWER - extracts.

Data from DATPAT pp 140 - 150, p 149
(Christmas tree problem, 10 cases, p 136)

next matrix 'A'

$$\begin{pmatrix} 0.756 & 0.216 \\ 0 & 0.756 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{matrix} 0.028 & 0.002 \\ 0.216 & 0.070 \\ 0.756 & 0.266 \\ 0 & 1 \end{matrix}$$

output for 'A', h=1, k=10 :-

0.569 0.325 0.088 0.017
0.000 0.569 0.325 0.105 X^2
0.000 0.000 0.569 0.431
0.000 0.000 0.000 1.000

0.430 0.368 0.153 (0.050) †
0.000 0.430 0.368 0.202 X^3
0.000 0.000 0.430 0.570
0.000 0.000 0.000 1.000

0.324 0.370 0.206 0.099
0.000 0.324 0.370 0.306 X^4
0.000 0.000 0.324 0.676
0.000 0.000 0.000 1.000

0.244 0.349 0.244 0.162
0.000 0.244 0.349 0.406 X^5
0.000 0.000 0.244 0.756
0.000 0.000 0.000 1.000

0.184 0.316 0.247 (0.233) †
0.000 0.144 0.316 0.499 X^6
0.000 0.000 0.184 0.816
0.000 0.000 0.000 1.000

0.139 0.276 0.274 0.308
0.000 0.139 0.276 0.582 X^7
0.000 0.000 0.139 0.861
0.000 0.000 0.000 1.000

0.105 0.240 0.271 0.384
0.000 0.105 0.240 0.655 X^8
0.000 0.000 0.105 0.895
0.000 0.000 0.000 1.000

0.079 0.204 0.259 (0.458) †
0.000 0.079 0.204 0.717 X^9
0.000 0.000 0.079 0.921
0.000 0.000 0.000 1.000

0.060 0.171 0.242 0.528
0.000 0.060 0.171 0.769 X^{10}
0.000 0.000 0.060 0.940
0.000 0.000 0.000 1.000

161 (cont'd p 162)

part of 'A', h=2, k=8 :-

0.060 0.171 0.242 0.528
0.000 0.060 0.171 0.769 X^{10}
0.000 0.000 0.060 0.940
0.000 0.000 0.000 1.000

0.034 0.117 0.198 0.651
0.000 0.034 0.117 0.849 X^{12}
0.000 0.000 0.034 0.966
0.000 0.000 0.000 1.000

0.019 0.077 0.154 0.749
0.000 0.019 0.077 0.905 X^{14}
0.000 0.000 0.019 0.981
0.000 0.000 0.000 1.000

0.011 0.050 0.115 0.824
0.000 0.011 0.050 0.939 X^{16}
0.000 0.000 0.011 0.989
0.000 0.000 0.000 1.000

output for 'C', k=5 :-

0.569 0.325 0.088 0.017
0.000 0.569 0.325 0.105 X^2
0.000 0.000 0.569 0.431
0.000 0.000 0.000 1.000

0.324 0.370 0.206 0.099
0.000 0.324 0.370 0.306 X^4
0.000 0.000 0.324 0.676
0.000 0.000 0.000 1.000

0.105 0.240 0.271 0.384
0.000 0.105 0.240 0.655 X^8
0.000 0.000 0.105 0.895
0.000 0.000 0.000 1.000

0.011 0.050 0.115 0.824
0.000 0.011 0.050 0.939 X^{16}
0.000 0.000 0.011 0.989
0.000 0.000 0.000 1.000

Output from MATPOWER - estimate.

Data for Christmas tree problem, 20 balls, (p 136) — see DATMAT files 300-350 (p 169)

Arithmetic sequence $A = 1$, $K = 10$.

$$X = \begin{pmatrix} 0.5693 & 0.3070 & 0.0795 & 0.0622 \\ 0 & 0.5693 & 0.3070 & 0.1237 \\ 0 & 0 & 0.5693 & 0.4307 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} 0.324 & 0.352 & 0.186 & 0.138 \\ 0 & 0.324 & 0.352 & 0.324 \\ 0 & 0 & 0.324 & 0.676 \\ 0 & 0 & 0 & 1 \end{matrix} \quad X^2$$

$$\begin{matrix} 0.185 & 0.300 & 0.340 & 0.275 \\ 0 & 0.185 & 0.300 & 0.515 \\ 0 & 0 & 0.185 & 0.815 \\ 0 & 0 & 0 & 1 \end{matrix} \quad X^3$$

$$\begin{matrix} 0.105 & 0.228 & 0.266 & 0.627 \\ 0 & 0.105 & 0.228 & 0.667 \\ 0 & 0 & 0.105 & 0.895 \\ 0 & 0 & 0 & 1 \end{matrix} \quad X^6$$

$$\begin{matrix} 0.060 & 0.162 & 0.217 & 0.560 \\ 0 & 0.060 & 0.162 & 0.778 \\ 0 & 0 & 0.060 & 0.940 \\ 0 & 0 & 0 & 1 \end{matrix} \quad X^5$$

etc. & X^{10}, which is:-

$$\begin{matrix} 0.006 & 0.019 & 0.052 & 0.925 \\ 0 & 0.006 & 0.019 & 0.977 \\ 0 & 0 & 0.006 & 0.994 \\ 0 & 0 & 0 & 1 \end{matrix} \quad (X^{10})$$

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Appendix I

Simulation of a Multiple-Server Shop

The following exercise is a manual simulation of a multiple-server shop, based on an idea from Unit 6 of the Open University Technology Course T341,(†) and is a much reduced version of the 'coal-depot' simulation given there on pp 66-99. This is an event-sequence simulation (see Section 3, p20, & the first version of the doctor's surgery simulation).

The description and instructions for carrying out the simulation are followed by a specimen record sheet to hold the movements of 20 customers, and a cut-out sheet for playing the 'game'. (Small slips representing customers, servers, labels for queues and serving points are provided. They are cut out and placed in appropriate positions on a table - "customers" are moved around from serving point to queue to serving point, as the rules prescribe). The activity was used in a practical session of a course given for teachers of Social Sciences, and the information provided, which is rather abbreviated, was supplemented orally. A few notes of amplification are therefore added here - they should be read in conjunction with p 171-2.

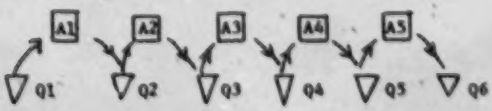
Queue discipline FIFO first in, first out - the usual rule for bus and shop queues

LIFO last in, first out - as for IN-trays

It is also possible that priority customers move to the head of the queue, or that the queue is served in random order.

Stages in setting up

1) In this problem, entities are customers - in other simulations they might be vehicles, ships, machines for repair, etc.
The life-cycle for customers in the initial model would be:



The diagram shows the location of queues and activity cards and the order in which customers pass through them. This represents (part of) the "board" on which the "game" is played - in later variants the arrow pattern would be altered.

Since servers are allocated one per counter we need not specify their life-cycles - but we do so in variants below.

(†) Ref: Jones, L et al. Technology: A third level course. Systems modelling, Unit 6; Simulation modelling, Open University 1975.

2) Activity cards always alternate (strictly) with queue cards, for uniformity (even if customers move at once from a 'queue' to the next activity).
3) Rules of the simulation include specification of the order in which activities - here, being served at successive counters - are performed, queue discipline, priorities if two activities compete for a queue, etc.
4) Customer cards are placed in position in the first queue.
5) Next-event-list holds time at which next move in simulation will be made. E.g. If customer at Meat will take 8 min to serve, at Drinks 5 min and at cash desk 3 min, and next customer in Q1 will be due to move to Q2 (Cake counter queue) in 7 min, the 'next event' is in 3 min, when cash desk service is completed.

Instructions

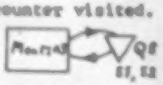
Each activity card carries information.

e.g. A1 gives - duration of this activity before moving to next queue. (This list of digits gives a random pattern of arrivals, spaced at intervals of 4,1,4,2.....minutes. When the list is exhausted it is used again from the beginning).
- source and destination of customers (here Q1 and Q2)
- items to be recorded on a record sheet (as customers enter and leave the activity)

If the model is changed, as in 'supermarket' and 'skipping' suggestions below, some information on activity cards may need to be altered - e.g. source and destination of customers (cards A3,A4,A5 have spaces for this).

Notes

Variants (c) Supermarket } Each customer card has symbols c.m.d. - circle
(d) Skipping } those of counters to be visited (decided randomly in "(d)") and cross out symbol when counter visited.

(e) Life cycle for a server on meat counter will be 

(Servers S1,S2, wait in a queue (Q8) until customer arrives, and retire to Q8 after service).

Similarly for A4,A3.

Finally, note that the variants suggested make the model more realistic or complex, and also show possibility of trying out other policies.
The case study given in T341 makes interesting reading and shows how the method was applied in a real-life situation (T341 pp 98-9).
Other case studies are also given there (New York fire fighting), and a reading list.

SIMULATION OF A MULTIPLE-SERVER SHOP

This simulation attempts to model the operation of a small shop containing 4 serving points, labelled CAKES, MEAT, DRINKS and CASH-DESK. A simple version is simulated first, then successive modifications are described which would model the behaviour of a motorway cafe or a supermarket.

AIMS 1) To illustrate how a simulation of a complex system can be set up and carried out.

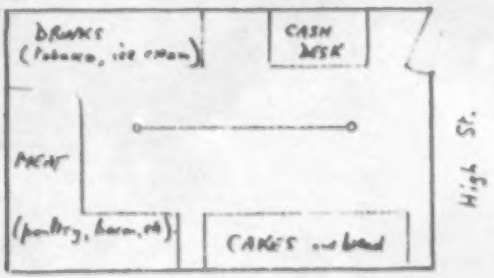
2) To show how the conditions of the simulation may be varied, either to produce a more accurate model, or to investigate the consequences of a proposed change in the system.

3) To indicate how by running a simulation, manually or on a computer, the behaviour of a complex system can be investigated, even when a mathematical approach becomes too difficult.

Description

Danesbury's is a small High St. shop built in 1965 with a floor area of 53 m^2. The owner, Fred Danesbury, is assisted in the shop by his wife, Gladys, and their two daughters Joan(23) and Susan(21). There are 3 counters : MEAT (Fred), CAKES (Joan) and DRINKS (Susan); Gladys, who is cheerful and slightly deaf, looks after the CASH-DESK. The shop is situated in a good position and was redecorated 3 years ago, when Fred introduced a one-way system, so that customers visit the 3 counters in turn, in the order CAKES, MEAT, DRINKS, before going to Gladys at the check-out. Fred has noticed that his customers invariably buy something at each counter, and that each one spends precisely 3 min, 4 min, and 2 min at the three respective counters, and 2 min at the CASH-DESK. They are less regular in their arrival at the shop, however; the time between arrivals averages just over 5 min - it can be as short as 1 min or as long as 12 min.

PLAN OF FRED'S SHOP



TASKS

1) Underline the information in the above passage which is relevant to performing a simulation.

2) Draw a simplified diagram of Fred's shop, representing it as a chain of 4 activities each preceded by a queue. (Activities are usually shown ▭ and queues ▽. Mark in the duration of each activity.)

3) What is the most likely rule of queue discipline in this case? What other rules are possible - in other situations?

4) Suggest modifications to this simplified model which will make it more realistic. (Some of these will be explored later).

STAGES IN SETTING UP THE SIMULATION

1) Draw out life-cycles for entities (customers), (see Fig.(2)).
2) Complete and place activity cards and queue cards.
3) Specify operating rules (eg. by numbering of activity cards).
4) Prepare record sheet.
5) Place entity (customer) cards.
6) Enter start-values on next-event list.

INSTRUCTIONS FOR CARRYING OUT THE SIMULATION

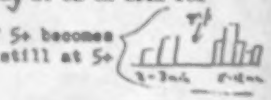
1) Scan through activities in increasing order.
   Start any appropriate activity by moving an entity (customer) from a queue.
   Update record sheet.
   Work out duration time and finish time - add to next-event list.
   Continue the scan (ie. from (1) above)
2) Ring time of next-event-due (end of an activity); advance to this time.
3) Move entities (customers) from activities into appropriate queues.
   Delete from next-event list
   Update record sheet.
   Repeat step (3) until complete;
   Then return to step (1)

NOTES

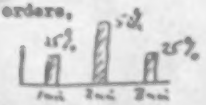
1) All queues are FIFO ; but could be LIFO, Priority, random.
   No limitations on queue size considered here - eg. crowded shop or discouraged customers.
2) VARIANTS:-
a) Initial model - small shop, 3 counters, + cash desk, visit all, in order; one server each = 4 staff.
b) Put MEAT first, interchanged with CAKE (This would probably cause more delay?)
c) Supermarket. Customers visit each counter, but in any order. (Record on customer card when completed). eg. "seeking shortest queue, else next activity, or "randomly."
d) Customers need not visit all counters (cf. motorway cafe:- cakes = sweet, meat = main course, drinks = tea/coffee) "Skipping" - mark counters-to-be-visited on customer card when new customer is generated.

e) As in a) but "redeploying" servers. Gladys to do DRINKS + CASH-DESK, extra server on MEAT. This implies that new entities ('servers') have to be followed through the system. New life-cycles for the 2 MEAT servers, (+ a queue) and for the DRINKS - CASH-DESK server, (+ another queue).
   (Add a note to Server card S4 :- "Server to give A5 priority over A4", customer gives usual priority ie. "A4 then A5")

f) Combine (c) and (d). Supermarket skipping, together with priority at cash desk for small orders.
g) Alter generating pattern of arrivals - eg. so that mean of 5+ becomes 7+ or 3+ ; or make arrivals more irregular, with mean still at 5+
h) Alter the generating pattern of service times, (see below).
i) Incorporate "server goes to lunch" into next-event list.

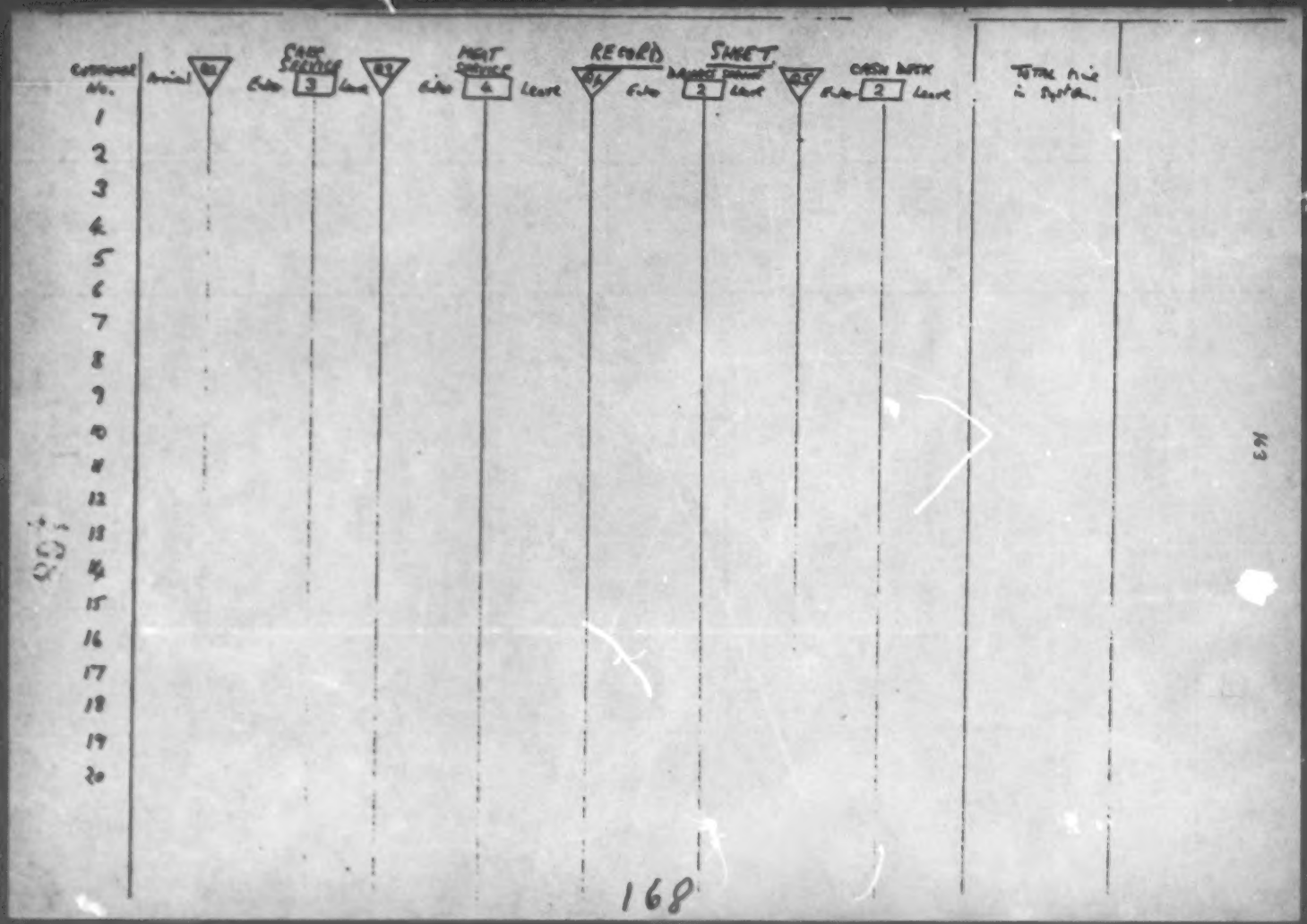


3) Measures of delay :- average waiting time,
   longest waiting time,
   convenience for customers with small orders,
   idleness of servers.



4) Service times could be generated from a distribution eg.

This could be achieved by tossing two coins { TT ——→ 1 min; HT or TH ——→ 2min; HH ——→ 3min


Customer No.
Arrival
Q2
Cash Service
Enter
3
Leave
Q3
Night Service
Enter
4
Leave
Record
Q4
Enter
Sheet
Balance Enquiry
2
Leave
Q5
Cash Back
Enter
2
Leave
Total time in System
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20

S4 | S1 | S2 | S3 | Shop simulation — phase

GENERATE ARRIVALS A1

DURATIONS (~ minutes):-

4	1	6	2	5	6	3	6	8	12	
7	4	1	7	4	5	4	5	5	3	
10	5	8	4	7	7	4	5	2	7	

Q1 ← customers → Q2

Record: Customer No. and Time OUT

C10 c – d | C0 c – d

C11 c – d | C1 c – d

Q4 — WAIT for DRINKS

CAKE COUNTER A2

DURATION 3 min.

Q2 → customers → Q3

Record:

IN: Customer no. and time | Customer no. and time OUT

C12 c – d | C2 c – d

C13 c – d | C3 c – d

Q5 — WAIT to PAY

Q3 — WAIT for MEAT

MEAT COUNTER A3

DURATION 4 min.

Q3 → customers → Q4

(→)

Record:

IN: Customer no. and time | Customer no. and (Sandwich) time OUT

C14 c – d | C4 c – d

C15 c – d | C5 c – d

Q6 — QUEUING

Q2 — WAIT for CAKE

DRINKS COUNTER A4

DURATION 2 min.

Q4 → customers → Q5

(→)

Record:-

IN: Customer no. and time | Customer no. and time OUT

C16 c – d | C6 c – d

C17 c – d | C7 c – d

Q7 — (CASH/DRINKS) WAIT for CUSTOMERS

Q1 — POOL of CUSTOMERS

CASH DESK A5

DURATION 2 min.

Q5 → customers → Q6

( )

Record:-

IN: Customer no. and time | Customer no. and time OUT

C18 c – d | C8 c – d

C19 c – d | C9 c – d

Q8 — (MEAT) WAIT for CUSTOMERS

Appendix II

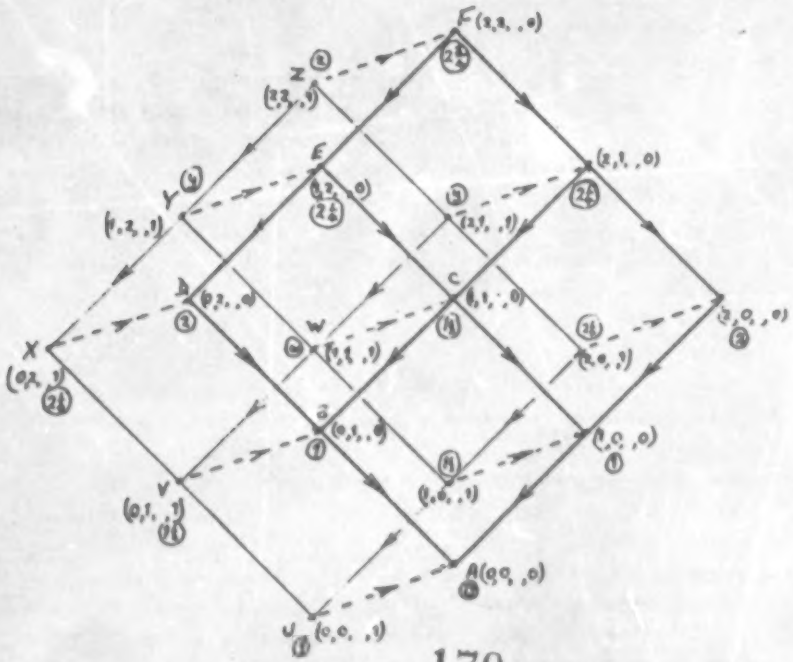
The Game of 'Beetle'

We have already considered this game (p 27) and other related waiting problems ('Octopus', p 29 and p 114); the collector's problem (problem C, p 1 also p 107); 'Lucky 7' (p 117); waiting time for '6', double '6', '5' and '6', '4', '5' and '6', (pp 114, 114), and for a 'run' of HHH (p 108).

Similar methods can be used, in principle, to find the expected number of throws needed to complete a game of Beetle.

We consider first the problem of collecting two eyes ('1's') and two feelers ('2's') together with a tail (a '4'). (We assume the head, '5', and body, '6', have already been collected, and, for the present ignore legs ('3's')). The states concerned may be represented by triples where (p,q, ,s) means p '1's', q '2's' and s '4's' remain to be collected.(†) The final target state is (0,0, ,0) and the initial state is (2,2, ,1). The states can be displayed as a 3 dimensional network as shown below, where the probability of a transition along any of the arcs is $\frac{1}{6}$.

(†) Notation agrees with p 116



Lines shown full ——— dotted..... } represent the acquisition of { either a '1' or a '2' / a '4'

The waiting time, (measured in 'units' of 6 'throws'), to reach (0,0, ,0) from each point is shown circled (compare p 115 , from which the waiting times for the 'top layer' ABCDEF of the diagram - states with labels (p,q, , 0) have been transferred).

By an argument similar to that of p 115,

Waiting time at W = w = $\frac{1\frac{1}{2} + 1\frac{1}{2} + 1\frac{1}{2}}{3} + \frac{1}{2}$

Average of wait once '1','2' or '4' appears + Expected wait until one of '1','2' or '4' appears

Thus w = $1\frac{5}{6}$ (p(1 or 2 or 4) = $\frac{1}{2}$, so expected wait = 2 'throws' = $\frac{1}{2}$ 'unit')

Similarly, waiting times for Y and Z are

$$y = \frac{2\frac{1}{2} + 2\frac{1}{2} + 1\frac{5}{6}}{3} + \frac{1}{2} = 2\frac{5}{9}$$

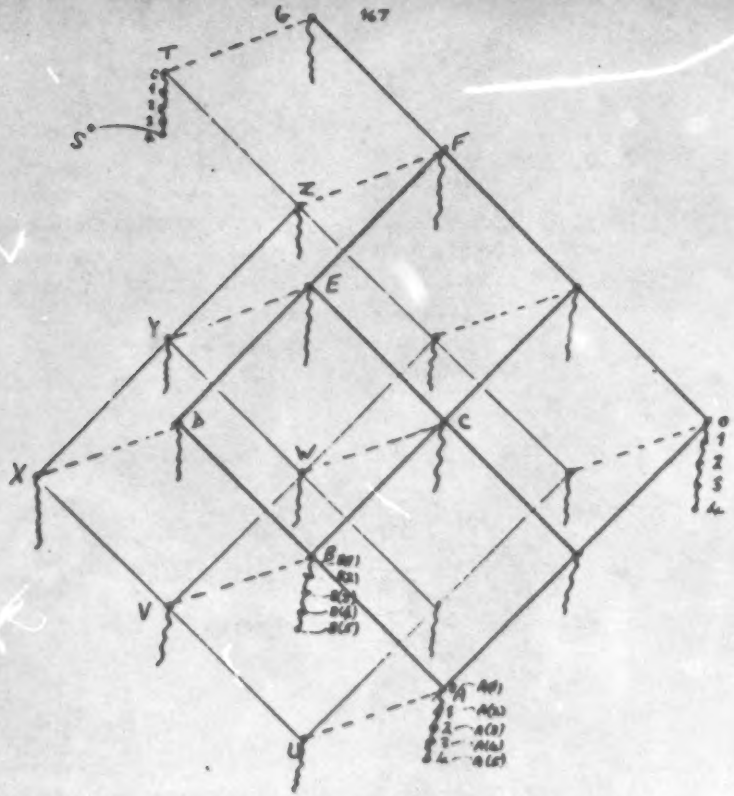
$$z = \frac{Y + y + 2\frac{1}{2}}{3} + \frac{1}{2} = 2.8796$$

Now consider the incorporation of legs ('3's') into the scheme. This associates with each existing state (p,q, ,s), four further states represented by (p,q,4,s), (p,q,3,s), (p,q,2,s) and (p,q,1,s), in which the number of legs still to be obtained are 4,3,2,1, respectively. (The earlier states could be relabelled (p,q,0,s) showing that all the legs had already been obtained). To represent this network a 4 dimensional 'picture' would be helpful. However we shall merely add a 'chain' of 4 states - with 4,3,2,1 legs still required - to each of the points in the previous network, but it should be remembered that, for each of the levels 0,1,2,3,4 of 'number of legs', a similar network of connections exists.

The states are labelled A —— F, U —— Z as shown, with suffixes J running from 1 to 5. (waiting times for the unlabelled states can be deduced by symmetry).

For example A(1) is used to denote the state (0,0,0,0), A(2) is (0,0,1,0), A(3) is (0,0,2,0) etc. B(1) is (0,1,0,0), V(1) is (0,1,0,1) and M(J) is (1,1,J-1,1) etc.

The relations given below may be written down - by a change of notation we use the same symbols to represent waiting times from the corresponding states. The first 12 of these are merely a recapitulation of earlier results in new notation, whilst the second group of 12 take into account the possibility of changes like (2,2,3,1) → (2,2,2,1) (i.e. Z(4) → Z(3)).



$A(1) = 0$

$B(1) = 1$

$C(1) = \frac{2 \cdot B(1)}{2} + \frac{1}{3} = 1\frac{1}{3}$

$D(1) = B(1) + 1 = 2$

$E(1) = \frac{C(1) + D(1)}{2} + \frac{1}{3} = 2\frac{1}{6}$

$F(1) = \frac{2E(1)}{2} + \frac{1}{3} = 2\frac{1}{6}$

$U(1) = 1$

$V(1) = \frac{B(1) + U(1)}{2} + \frac{1}{3} = 1\frac{1}{3}$

$W(1) = \frac{2V(1) + C(1)}{3} + \frac{1}{3} = 1\frac{5}{6} (= u)$

$X(1) = \frac{V(1) + D(1)}{2} + \frac{1}{3} = 2\frac{1}{6}$

$Y(1) = \frac{X(1) + E(1) + W(1)}{3} = 2\frac{1}{6} (= y)$

$Z(1) = \frac{2Y(1) + F(1)}{3} + \frac{1}{3} = 2.8796 (= z)$

$A(J) = J-1$

$B(J) = \frac{B(J-1) + A(J)}{2} + \frac{1}{3}$

$C(J) = \frac{2B(J) + C(J-1)}{3} + \frac{1}{3}$

$D(J) = \frac{D(J-1) + B(J)}{2} + \frac{1}{3}$

$E(J) = \frac{D(J) + C(J) + E(J-1)}{3} + \frac{1}{3}$

$F(J) = \frac{2E(J) + F(J-1)}{3} + \frac{1}{3}$

$U(J) = \frac{A(J) + U(J-1)}{2} + \frac{1}{3}$

$V(J) = \frac{B(J) + U(J) + V(J-1)}{3} + \frac{1}{3}$

$W(J) = \frac{2V(J) + C(J) + W(J-1)}{4} + \frac{1}{2}$

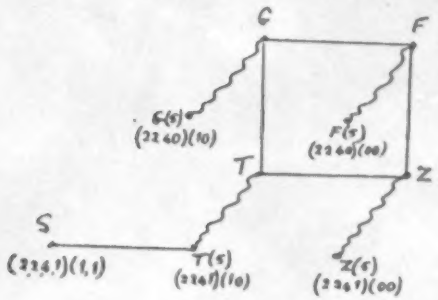
$X(J) = \frac{V(J) + D(J) + X(J-1)}{3} + \frac{1}{3}$

$Y(J) = \frac{X(J) + W(J) + E(J) + Y(J-1)}{4} + \frac{1}{2}$

$Z(J) = \frac{2Y(J) + F(J) + Z(J-1)}{3} + \frac{1}{2}$

The computer program BEETLW and print-out given on p 174 illustrate the calculation of the waiting times - the program also includes calculations for the other possible states displayed in the diagram below. S is the starting state, represented by (2,2,4,1) (1,1) - no 'component parts' of the beetle have been collected and the only possible transition is to T(5) (by throwing a '6'). At T(5), labelled (2,2,4,1) (1,0), collection of 'legs' ('3's'), tail ('4's') and head ('5's') can begin, represented by

$T(J) \to T(J-1)$
$T(J) \to G(J)$
$T(J) \to Z(J)$
} respectively



The additional relations are

$G(1) = F(1) + 1$

$T(1) = \frac{G(1) + Z(1)}{2} + \frac{1}{2}$

$G(J) = \frac{F(J) + G(J-1)}{2} + \frac{1}{2}$

$T(J) = \frac{G(J) + Z(J) + T(J-1)}{3} + \frac{1}{2}$

$S = 1 + T(5)$

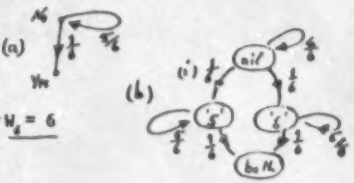
Note. If J takes values from 1 to 5, the value of S, printed out at the end, is the expected wait to obtain 'Beetle' ("4 legged version") - the corresponding value for "6-legged" Beetle is obtained by extending the loop so that J runs from 1 to 7.

From the computer output, expected wait is 5.9 units (for '4-legged' Beetle - and 8 units for '6-legged' Beetle).

Since the wait at (0040) (00) is 4 units, and the wait for the initial '6' (01) is 1 unit, the additional wait for head, eyes, feelers and tail is about 1 unit. (We might argue intuitively, that by the time body, head and 4 legs have been acquired, the remaining components are 'likely' to have been 'picked up en route'; just as the process of acquiring 2 eyes and 2 feelers requires an additional (average) wait of 1.9 - 2.8 if all legs have been obtained and only about 0.5 if no legs have been obtained: $F(1) - A(1) = 2.75$, $Z(1) - U(1) = 1.88$; c.f. $F(5) - A(5) = 0.49$, $Z(5) - U(5) = 0.45$.

Reconsidering the FIV6 Problem

On pp 108 and 115, two essentially different arguments were used to calculate waiting times; fortunately they yielded the same answer! As an illustration, consider the simple cases of (a) 'waiting for a '6'', and (b) 'waiting for '5' and '6'' with state diagrams as shown:-

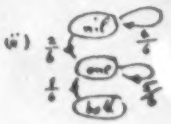


If W_6 is the wait time for a single '6', the argument of p108 runs:-

$$W_6 = \frac{1}{6}(1+0) + \frac{5}{6}(1+W_6) \quad \text{i.e. } W_6 = 1 + \frac{5}{6}W_6 \Rightarrow \underline{W_6 = 6}$$

outcome '6' — outcome 'not 6'

A similar argument gives (fig. b(i)):



$$W_{5,6} = \frac{1}{6}(1+W_5) + \frac{1}{6}(1+W_6) + \frac{4}{6}(1+W_{5,6})$$

and since $W_5 = W_6 = 6$, from above,

$$W_{5,6} = 1 + 1 + 1 + \frac{2}{3}W_{5,6} \quad \text{i.e. } \underline{\underline{W_{5,6} = 9}}$$

The argument of p 115 uses fig b(ii):-

Expected wait to reach 'one' is $\frac{6}{2} = 3$

Expected wait once move has taken place is $\frac{6+6}{2}$ (average of 2 equally likely outcomes)

$= 6$; $\underline{\underline{\text{Total wait} = 9}}$ (as before).

The table of values of $\frac{1}{6}W_{p,q}$ (from pp 115,116) is:-

p ↓ / q →	0	1	2	3	4
0		1	2	3	4
1	1	$1\frac{1}{3}$	$2\frac{1}{3}$	$3\frac{1}{3}$	$4\frac{1}{3}$
2			$2\frac{1}{3}$	$3\frac{2}{3}$	$4\frac{1}{3}$
3				$3\frac{16}{33}$	$4\frac{17}{33}$
4					$5\frac{2}{33}$

Although some explicit results can be obtained, e.g. $W_{1,q} = 6\left(q + \frac{1}{2^q}\right)$

$$W_{2,q} = 6\left(\frac{2}{3} + \frac{q+4}{2^{q+2}}\right),$$

there seems to be no general pattern, and the best method of calculating such wait times seems to be the use of the recurrence relations, (cf. BEETLW),

in this case: $W_{pq} = \frac{W_{p,q-1} + W_{p-1,q}}{2} + 3$ with $W_{0,q} = 6q$

General Collecting Problems

Beetle may be viewed as one of a general series of 'collecting' problems, of which simple examples are "waiting for a '6'" - i.e. achieving a target (000001), p^{10^7}; collecting N '6's' (target (00000N); and collecting a set of n objects, target $(\underbrace{11\ldots11}_{n})$, p^{10^7}. In Beetle, as well as collecting multiple objects simultaneously (4 legs and head....), we are subject to certain restrictions on order - e.g. '6' for a body must be collected first; eyes and feelers ('1's' and '2's') can only be collected after a head ('5') has been obtained. This introduces additional complications. All the cases so far considered relate to collecting A&B&C....where A,B,C are dependent events (e.g. the occurrence of a '5' automatically excludes the occurrence of a '6'). For the simple case of collecting 2 'events' A,B we have four possibilities. The table below refers to the particular case of the events

A - throw '5' with a die

B - throw '6' with a die,

and the figures in the table represent the expected waiting times in 'throws'. (For independent events we could use 2 dice).

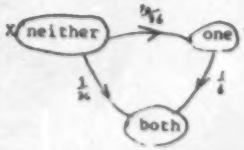
	A&B	A or B
A,B are dependent →	9	3
independent →	8.73	3.27

Discussion:-

Dependent events. prob(A or B) = $\frac{1}{3}$; W=3

Independent events. prob(A or B) = $\frac{11}{36}$

$W = \frac{36}{11} = 3.27$



$W_Y = 6$, $W_X = \frac{36}{11} + \frac{1}{11}(0) + \frac{10}{11}(6) = \frac{96}{11} = 8.73$

↑ wait to move

Theoretical analysis of waiting problems of this type is rather complicated, though they are fairly easy to simulate on a computer.

Some similar situations are (see Watson, F.R., 1981)

(a) Parents decide to stop having children when they have at least 3 boys and at least 3 girls. What is the expected family size? (dependent)

(b) For a medical experiment I need a sample of each of blood groups A,B AB and O. How many individuals must I test? (dependent)

(c) Computer components fail with very low probability in a period of 1 sec. (Prob? of more than one failure is negligible). Because of inbuilt redundancy, operation will continue, but only until either 3 of type A or 3 of type B components have failed.
What is the expected life? (independent, though we treat as if dependent because of ‡).

An Alternative Approach

An alternative method of attack on the Beetle problem is to use Markov chain methods as in Section 8, where the simple collection problems '$5,6^2$' (or (0000,1,2) in our notation) and '4,5,6', (i.e. (000,1,1,1)) were considered (p132, p137). We consider here the 'Octopus' problem, ('$1^4, 2^4$' - i.e. (4,4,0000) - see p29 - evidently this "includes" the $5,6^2$ problem). This may be simulated using either of the programs FIV6 or COLLGEN. The MARKOV solution is given below.

As before, let each state be labelled (a,b) where a '1's' and b '2's' remain to be collected. We reduce the overall number of states - and so the dimensions of the matrices involved - by using symmetry, pairing the states (a,b) and (b,a) - i.e. we consider only states with a≥b and adjust the transition probabilities accordingly. (In general this reduces n^2 states to $n + \frac{n^2-n}{2}$ i.e. to $\frac{n^2+n}{2}$).

The associated matrices (zero entries are omitted) were calculated by hand - the computer output which follows gives a check.

$$
\begin{array}{c|cccccccccccc}
\text{from}\rightarrow\ \text{to}\downarrow & 44 & 43 & 42 & 33 & 41 & 32 & 40 & 31 & 22 & 30 & 21 & 20 & 11 & 10 & 00 \\
44 & 4 & 2 & & & & & & & & & & & & & \\
43 & & 4 & 1 & 1 & & & & & & & & & & & \\
42 & & & 4 & 0 & 1 & 1 & & & & & & & & & \\
33 & & & & 4 & 0 & 2 & & & & & & & & & \\
41 & & & & & 4 & 0 & 1 & 1 & & & & & & & \\
32 & & & & & & 4 & 0 & 1 & 1 & & & & & & \\
40 & & & & & & & 5 & 0 & 0 & 1 & & & & & \\
31 & & & & & & & & 4 & 0 & 1 & 1 & & & & \\
22 & & & & & & & & & 4 & 0 & 2 & & & & \\
30 & & & & & & & & & & 5 & 0 & 1 & & & \\
21 & & & & & & & & & & & 4 & 1 & 1 & & \\
20 & & & & & & & & & & & & 5 & 0 & 1 & \\
11 & & & & & & & & & & & & & 4 & 2 & \\
10 & & & & & & & & & & & & & & 5 & 1 \\
00 & & & & & & & & & & & & & & & 6
\end{array}
$$

$$
\frac{1}{6}N = \begin{bmatrix}
\frac{1}{1} & & & & & & & & & & & & & & & \\
 & \frac{1}{4} & & & & & & & & & & & & & & \\
 & & \frac{1}{4} & & & & & & & & & & & & & \\
 & & & \frac{1}{4} & \frac{1}{4} & & & & & & & & & & & \\
 & & & & & \frac{1}{4} & \frac{1}{4} & & & & & & & & & \\
 & & & & & & \frac{1}{4} & \frac{1}{4} & & & & & & & & \\
 & & & & & & & \frac{1}{5} & \frac{1}{4} & \frac{1}{4} & & & & & & \\
 & & & & & & & & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & & & & & \\
 & & & & & & & & & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & & & & \\
 & & & & & & & & & & \frac{1}{5} & \frac{1}{4} & \frac{1}{4} & & & \\
 & & & & & & & & & & & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & & \\
 & & & & & & & & & & & & \frac{1}{5} & \frac{1}{4} & \frac{1}{4} & \\
 & & & & & & & & & & & & & \frac{1}{4} & \frac{1}{2} & \\
 & & & & & & & & & & & & & & \frac{1}{5} & \frac{1}{1} \\
 & & & & & & & & & & & & & & & \frac{1}{6}
\end{bmatrix}
$$

Extracts from output :- (data for 'Octopus' in file DATMAT, p 49).

FUNDL MATRIX (i.e. $N = (I-Q)^{-1}$).

3.00	3.00	1.50	1.50	0.75	2.25	0.75	1.50	1.12	2.25	1.87	4.12	0.94	6.00
0.00	3.00	1.50	1.50	0.75	2.25	0.75	1.50	1.12	2.25	1.87	4.12	0.94	6.00
0.00	0.00	3.00	0.00	1.50	1.50	1.50	1.50	0.75	3.00	1.50	4.50	0.75	6.00
0.00	0.00	0.00	3.00	0.00	3.00	0.00	1.50	1.50	1.50	2.25	3.75	1.12	6.00
0.00	0.00	0.00	0.00	3.00	0.00	3.00	1.50	0.00	4.50	0.75	5.25	0.37	6.00
0.00	0.00	0.00	0.00	0.00	3.00	0.00	1.50	1.50	1.50	2.25	3.75	1.12	6.00
0.00	0.00	0.00	0.00	0.00	0.00	6.00	0.00	0.00	6.00	0.00	6.00	0.00	6.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	3.00	0.00	3.00	1.50	4.50	0.75	6.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	3.00	0.00	3.00	3.00	1.50	6.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	6.00	0.00	6.00	0.00	6.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	3.00	3.00	1.50	6.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	6.00	0.00	6.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	3.00	6.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	6.00

RESULTS OF 10 SIMULATIONS :-
MEAN NO OF TIMES EACH STATE

2.90	3.30	1.90	1.00	0.30	3.40	0.80	1.40	0.40	3.40	2.20	6.20	0.70	7.10
0.00	0.90	1.60	1.80	1.30	1.40	0.90	0.50	1.80	2.10	1.20	4.30	0.40	2.50
0.00	0.00	2.20	0.00	1.30	1.10	2.60	0.70	1.20	5.30	1.50	4.20	0.20	5.50
0.00	0.00	0.00	1.20	0.00	2.80	0.00	1.90	1.20	1.00	1.90	3.90	0.90	4.30
0.00	0.00	0.00	0.00	2.20	0.00	3.30	1.70	0.00	3.30	0.10	7.80	0.00	3.60
0.00	0.00	0.00	0.00	0.00	4.20	0.00	1.50	1.10	0.00	2.30	1.40	2.40	6.00
0.00	0.00	0.00	0.00	0.00	0.00	6.40	0.00	0.00	5.80	0.00	6.40	0.00	7.20
0.00	0.00	0.00	0.00	0.00	0.00	0.00	3.70	0.00	1.60	1.80	2.50	1.00	7.90
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.20	0.00	2.20	2.70	1.50	6.30
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	4.70	0.00	9.70	0.00	5.60
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.70	1.00	1.70	4.90
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	3.40	0.00	7.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	2.20	4.40
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	4.50

MEAN NO OF STEPS TO ABSORPTION (Theoretical results in brackets).

36.0	.7	26.8	20.1	23.0	19.7	26.8	19.5	16.9	21.0	10.3	11.4	7.(	5.5
(30.56	23.56	25.6	22.62	26.37	20.42	24	19.25	16.5	18	12.5	12	9	6)

To solve the Beetle problem in this way would involve very large matrices, however: the total number of states is 101, $\left[\underset{\text{body}}{1} + \underset{\text{legs}}{5} \times 2 \times \underset{\text{tail}}{(1+3\times 3)}\right]$, or 71 if grouping is used, as in BEETLW. $\left[1 + 5 \times 2 \times (1 + \underset{\text{eyes/feelers}}{6})\right]$.

Note, though, that this method gives more information than the earlier approach (p 168), since MARKOV or MATDN could be used to give the distribution of waiting times, as well as the mean (expected) waiting time for Beetle. As in many situations of interest, simulation (cf. BEETLE and BEETLH) provides a simple, and arguably more effective, approach.

Concluding remarks

The Markov method - or simulation - would give the distribution of lengths of Beetle games. By simple considerations we can see that:-

Minimum possible length of a game is 11 throws (instant success)

Expected length of a game lies between 30 (expected wait for '6' and four '3's')

and 66 (sum of expected waits for all components

Computer simulation (using BEETLH) gave the following distribution:-

```
HOW MANY REPETITIONS?                    (Extracts from output)
?                                                  Game Length (total) for two games
 100
 45  30  24  27  31  26  32  22  41  73  48  44  37  25  34  26  27  31
 48  22  57  35  26  31  21  37  54  19  49  35  33  37  31  37  29  33
 54  31  24  25  26  45  44  47  27  30  25  41  35  33  40  32  26  36
 27  26  37  57  59  37  33  31  37  30  24  32  30  35  26  30  26  29
 41  53  38  45  40  40  44  31  25  25  21  52  27  24  45  34  34  24
 20  45  27  26  46  32  36  31  36  29  31  35  41  55  44  35  21  34
 31  42  24  37  54  24  25  14  39  39  27  24  43  35  23  39  31  22
 47  23  25  34  51  39  39  51  41  33  45  30  27  37  20  25  29  31
 37  52  26  36  25  30  32  21  38  41  31  36  35  22  40  47  79  54
 31  29  35  30  40  29  25  34  40  35  25  21  38  34  42  28  32  29
 36  34  28  28  28  45  50  62  33  54  43  39  27  17  54  57  29  31
 34  30  25  24  31  21  34  46  55  46  37  31  51  48  18  53  37  26
 60  32  33  49  26  34  41  51  24  17  17  25  34  33  24  40  39  49
 42  32  30  19  31  29  31  25  37  30  28  21  22  52  49  35  32  34
 17  43  40  25  35  48  38  51  43  31  42  32  31  37  51  16  31  38
 29  54  28  42  48  51  23  43  39  34  53  31  35  29  25  26  37  35
 30  30  32  20  25  19  24  38  38  53  62  31
AVERAGE (TOTAL) GAME LENGTH WAS  35.24333
RESULTS FOR 2ND HALVES OF GAMES (IE. AFTER '6'):-
HISTOGRAM OF RESULTS
```

Two other problems now suggest themselves:-

(i) Beetle is usually played with N tables of 4 - a winner emerges at any table as soon as one contestant obtains a 'full set'. What is the expected wait for Beetle in a game of 4N people?

(ii) How far am I likey to have 'progressed through the network' when "Beetle!" is called? To particularise, suppose I score points, equal to the sum of the numbers of the Beetle components I have collected (e.g. 6 body, 3 each leg etc.). What is my expected score - and the distribution of my score - when "Beetle!" is called?

These are considered in the article referred to above - Watson, F.R. (1981), "How long is a game of Beetle?"

Calculation of waiting times for BEETLE problem (p%f)
using program BEETLW

A)	0.00	1.00	2.00	3.00	4.00	5.00	6.00
B)	1.00	1.50	2.25	3.13	4.06	5.03	6.02
C)	1.50	1.83	2.44	3.23	4.12	5.06	6.03
D)	2.00	2.25	2.75	3.44	4.25	5.14	6.08
E)	2.25	2.44	2.88	3.52	4.30	5.17	6.09
F)	2.75	2.88	3.21	3.75	4.45	5.26	6.15
G)	1.00	1.50	2.25	3.13	4.06	5.03	6.02
V)	1.50	1.83	2.44	3.23	4.12	5.06	6.03
W)	1.83	2.08	2.60	3.32	4.17	5.09	6.04
X)	2.25	2.44	2.88	3.52	4.30	5.17	6.09
Y)	2.44	2.60	2.99	3.59	4.34	5.19	6.10
Z)	2.88	2.99	3.30	3.81	4.48	5.28	6.16
6)	3.75	3.81	4.01	4.28	4.91	5.59	6.27
7)	3.81	3.87	4.06	4.42	4.94	5.60	6.30

7.375462 '=EXPECTED WAIT FOR BEETLE

```
0010 K=7
0020 A(1)=0
0030 B(1)=1
0040 C(1)=(2*B(1)+1)/2
0050 D(1)=2
0060 E(1)=(C(1)+D(1)+1)/3
0070 F(1)=(2*E(1)+1)/2
0080 U(1)=1
0090 V(1)=(B(1)+U(1)+1)/2
0100 W(1)=(2*V(1)+C(1)+1)/3
0110 X(1)=(V(1)+D(1)+1)/2
0120 Y(1)=(X(1)+E(1)+W(1)+1)/3
0130 Z(1)=(2*Y(1)+F(1)+1)/3
0140 G(1)=F(1)+1
0150 T(1)=(G(1)+Z(1)+1)/2
0160 FOR J=2 TO K
0170 A(J)=J-1
0180 B(J)=(B(J-1)+A(J)+1)/2
0190 C(J)=(2*B(J)+C(J-1)+1)/3
0200 D(J)=(D(J-1)+B(J)+1)/2
0210 E(J)=(D(J)+C(J)+E(J-1)+1)/3
0220 F(J)=(2*E(J)+F(J-1)+1)/3
0230 U(J)=(A(J)+U(J-1)+1)/2
0240 V(J)=(B(J)+U(J)+V(J-1)+1)/3
0250 W(J)=(2*V(J)+C(J)+W(J-1)+1)/4
0260 X(J)=(V(J)+D(J)+X(J-1)+1)/3
0270 Y(J)=(X(J)+W(J)+E(J)+Y(J-1)+1)/4
0280 Z(J)=(2*Y(J)+F(J)+Z(J-1)+1)/4
0290 G(J)=(G(J-1)+F(J)+1)/2
0300 T(J)=(G(J)+Z(J)+T(J-1)+1)/3
0310 NEXT J
0320 S=1+T(K)
0330 MATPRINT USING "10.20,20", A;B;C;D;E;F;U;V;W;X;Y;Z;G;T;
0340 PRINT S;" =EXPECTED  WAIT FOR BEETLE"
0350 END
```

Appendix III

Simulation Methods in Statistics

Computer programs and simulations can be used in the teaching of statistical concepts, from simple investigations of the effect on mean and variance, of transformations like $x' = x + K$, $x' = Kx$, through comparison of Normal and Binomial distributions, to more complex ideas such as sampling theory (sampling distributions of such statistics as means, variance, etc.) Theoretical predictions such as the distribution, in samples of size n, of $\frac{\bar{x} - \mu}{\sigma}\sqrt{n}$ (approximately standard Normal, by the Central Limit theorem), may be investigated, and the behaviour for different parent populations and varying sample sizes may be examined. Although experimental verification of these results ought to be unnecessary, given a mathematical proof, experience suggests that numerical demonstration is helpful both in 'bringing to life' a theoretical abstraction and also in providing experience of the variability of results and the degree of departure from prediction which may be expected. Distribution of sample variance can lead to the experimental generation of the χ^2 and F distributions, and comparison of the results with theory. We give here just one illustration - an investigation of the Central Limit theorem; further examples may be found in the references given below.

The BASIC program SAMPLER, which is listed below, produces K samples of size N, where K and N are input, from one of three populations (uniform, Normal, or "triangular" - the latter with frequency distribution as in fig 1). The mean and standard deviation of each sample is calculated. Using another program DISPLAY (not given here) applied to output from SAMPLER, we can produce a histogram showing the distribution of means of samples - with the mean and variance of this distribution - for comparison with theoretical predictions.

fig 1

References

Holmes, P.M. Using computers in the teaching of statistics, Math. Gazette, 410 Dec 1979, pp228-46

Watson, F.R. Suggestions for teaching sixth form mathematics with a computer (Part II), Institute of Education, University of Keele, 1980

Watson, F.R. Statistical simulations using a computer, Statistical Education, (in press).

SAMPLES

```
0010 RANDOMISE
0020 REM GIVES K SAMPLES OFSIZE N FROM ONE OF 3 DISTNS
0030 PRINT "THIS PROGRAM ALLOWS YOU TO CHOOSE SEVERAL SAMPLES OF "
0040 PRINT "A CHOSEN SIZE FROM ONE OF THREE DISTRIBUTIONS"
0050 PRINT "IT THEN PRINTS OUT THE SAMPLES, WITH THE MEAN AND S.D. OF EACH"

0060 PRINT
0070 PRINT
0080 PRINT "TYPE NAME OF FILE TO WHICH DATA TO BE OUTPUT"
0090 PRINT "FOR VDU TYPE      "    "
0100 INPUT F$
0110 FILES #1,F$
0120 PRINT "TYPE CHOICE OF DISTRIBUTION"
0130 PRINT "N  FOR NORMAL"
0140 PRINT "U FOR UNIFORM"
0150 PRINT  "T FOR TRIANGULAR"
0160 PRINT "TYPE CORRESPONDING LETTER"
0170 INPUT A$
0180 IF A$="N" THEN C=1:GOSO 220
0190 IF A$="U" THEN C=2 :GOTO 220
0200 IF A$="T" THEN C=3 : GOTO 220
0210 PRINT "ONLY THESE THREE DISTRIBUTIONS AVAILABLE":GOTO 130
0220 PRINT "TYPE VALUES OF POPULATION MEAN AND SD."
0230 PRINT "ON THE SAME LINE WITH A COMMA BETWEEN"
0240 INPUT M,S
0250 PRINT "TYPE NUMBER OF SAMPLES"
0260 INPUT K
0270 PRINT "TYPE SIZE OF EACH SAMPLE"
0280
0290 INPUT N
0300 PRINT "     ******* RESULTS ********"
0310 PRINT #1,"THERE ARE";K;" ROWS"
0320 PRINT #1,"EACH ROW IS A SAMPLE OF SIZE",N,"FROM THE POPULATION SPECIF
IED"
0330 IF F$="" THEN 360
0340 PRINT #1,"BY";A$;"MEAN=";M;"STANDARD DEVN =";S
0350 PRINT #1,"TWO FIGS ON EACH INTERMEDIATE ROW ARE MEAN AND SD OF SAMPLE"

0360 FOR I=1 TO K
0370 M1=0:S1=0
0380 FOR J=1 TO N
0390 ON C GOSUB 510,590,670
0400
0410 M1=M1 +X
0420 S1=S1+X^2
0430 PRINT USING "#0.20",#1,X;
0440 NEXT J
0450 M1=M1/N
0460 S1=SQR(S1/N-M1^M1)
0470 PRINT #1,
0480 PRINT USING "#0.20", #1,M1,S1 IF F$="" THEN PRINT "  MEAN AND S.D."
0490 NEXT I
0500 GOTO 640
```

(contd. opposite)

```
0510 X=0
0520 FOR P=1 TO 8
0530 X1=RND
0540 X=X+X1
0550 NEXT P
0560 X=(X-4)*SQR(1.5)
0570 X=M+X*S
0580 RETURN
0590 X=M+(2*RND-1)*S*SQR(3)
0600 RETURN
0610 X=SQR(16*RND)-SQR(8)
0620 X=M+X*S
0630 RETURN
0640 END
```

Variables

FS file name for output
AS type of distribution
M,S mean and s.d. of distribution
K no of samples, control word 6 I
N size of sample, control word 7 I
M1, S1 ΣX, ΣX^2
X single observation from parent population
P control variable = 510 520

Subroutines :- 510 - 630 Normal
590 - 600 Uniform
610 - 630 Triangular

Example of output :-

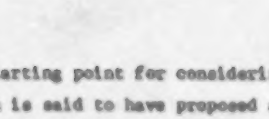
```
THIS PROGRAM ALLOWS YOU TO CHOOSE SEVERAL SAMPLES OF
A CHOSEN SIZE FROM ONE OF THREE DISTRIBUTIONS
IT THEN PRINTS OUT THE SAMPLES, WITH THE MEAN AND S.D. OF EACH


TYPE NAME OF FILE TO WHICH DATA TO BE OUTPUT
FOR YOU TYPE      ?
fp
TYPE CHOICE OF DISTRIBUTION
N  FOR NORMAL
U FOR UNIFORM
T FOR TRIANGULAR
TYPE CORRESPONDING LETTER
fn
TYPE VALUES OF POPULATION MEAN AND SD.
ON THE SAME LINE WITH A COMMA BETWEEN
?50,10
TYPE NUMBER OF SAMPLES
?4
TYPE SIZE OF EACH SAMPLE
?10
       ******** RESULTS ********
THERE ARE 4  ROWS
EACH ROW IS A SAMPLE OF SIZE      10
FROM THE POPULATION SPECIFIED


72.97  23.39  73.12  48.13  53.11  43.36  49.13  44.34  61.65  68.23
53.62  14.88 MEAN AND SD
47.98  55.23  58.49  64.79  52.04  53.10  57.27  53.72  56.77  60.31
55.95   4.42 MEAN AND SD
37.67  58.19  68.52  66.57  47.09  48.74  61.31  55.69  44.58  68.13
53.23  10.23 MEAN AND SD
54.56  42.45  57.49  63.88  49.39  58.97  45.58  58.40  48.28  52.29
53.23   6.33 MEAN AND SD
```

Appendix IV

The Efficiency of Simulation

The Buffon needle problem provides a starting point for considering problems of efficiency in simulation. Count Buffon is said to have proposed an experiment which can be used to give an estimate of π.

A needle of length L is dropped repeatedly on a floor ruled with parallel lines a distance d apart. ($L < d$).



If the needle comes to rest lying across one of the lines, we call this a "success". The probability of success is $\frac{2L}{\pi d}$.

(T.H. O'Beirne† gives an interesting account of Lazzarini's experimental determination of π as 3.1415929 ($\frac{L}{d} = \frac{5}{6}$; 1808 successes in 3408 throws), with an explanation of how such a high degree of accuracy (almost certainly fraudulent!) might have been achieved).

A typical experimental result might be, that 306 throws are needed (with $d = 2L$) to achieve 100 successes. This gives an 'observed' value for π of 3.06, which almost all pupils will find singularly unconvincing! The problem is that though the long run probability of success is indeed $\frac{1}{\pi}$, the variability of estimates based on even 100 experiments is large, and even with 10^6 throws we might get only 2 or 3 decimal place accuracy.

The same processes are at work in all simulations - thus successive simulations of a doctor's surgery might give average patient waiting times of 5.1, 7.2, 8.7, 9.1, 6.4, 9.3 - indicating a mean waiting time of 7.63 min. with standard deviation 1.68. How accurate is the estimate, 7.63 ?

Accuracy can generally be improved by performing further simulations and incorporating the additional data, but we can use the result (p175) that the mean of a sample of N observations is distributed about the population mean μ with variance $\frac{\sigma^2}{N}$. Thus in addition to giving an estimate for μ of we can compute a "95% confidence interval" for μ (that is, an interval determined by a method which, in a large number of calculations, will enclose the mean μ with probability 0.95). For the Normal distribution ($\mu-2\sigma$, $\mu+2\sigma$) gives an approximate 95% confidence interval. We do not know σ (but only "s", the sample s.d.) - it can be shown that the t-distribution is appropriate here, and that a 95% confidence interval is $(7.63 - 2.57 \times \frac{1.68}{10}, 7.63 + 2.57 \times \frac{1.68}{10})$ i.e. 7.20 - 8.06. The issue here is that the degree of precision of the estimate is

(†) O'Beirne, T.H. Puzzles and paradoxes, OUP, 1965

known - this is important, because although an estimate may be close to the true value, this does not help us unless we know how close it is likely to be. (Put simply, it may be better to have a large error which is known than an error which is actually much smaller, but is not known to be so.)

It is a major objective in serious work on simulation to devise ways of reducing the variance of estimates, thus increasing their precision. In a complex situation, a single run may involve many processes and much computation; merely increasing the number of runs may be an unacceptable way of ensuring greater accuracy. The methods used in practice to reduce variance and estimate error can only be mentioned here - further details may be found in Smith (1968) or in Hammersley and Handscomb (1964). They include:-

(i) Using random numbers in such a way that samples are deliberately drawn from opposite ends of distributions in two successive runs ("antithetic variables") - the mean of two estimates obtained in this way is likely to be much more accurate than the mean of two independent runs.

(ii) Estimating the error by applying the same simulation, using the same sequence of pseudo-random numbers, to a similar but simpler problem which can be solved analytically; the error in the known model indicates the approximate size of the error in the more complex situation.

(iii) Comparing different values of parameters, or different alternative strategies, by simulating each situation using the same (reproducible) set of random numbers (chance fluctuations will then affect the outcome similarly in all cases).

(see Smith 1968, pp62-63).

We conclude with some illustrations given by Hammersley and Handscomb (1964)† which exemplify very clearly the point made earlier (p 51) about "taking thought".

(A) Evaluating Integrals

Suppose $0 \leq f(x) \leq 1$ for $0 \leq x \leq 1$.

To find $I = \int_0^1 f(x)dx$ we can :

either (i) choose points (r_1, r_2) at random in the unit square

If $r_2 > f(r_1)$ score a 'miss', otherwise a 'hit'. The proportion of 'hits' gives the area under the curve.

or (ii) estimate I by $\frac{1}{n}(f(r_1) + f(r_2) + \ldots + f(r_n))$ where $r_1, r_2, \ldots r_n$ are random numbers between 0 and 1.

† This book is not elementary - but it makes fascinating reading, and gives insights into the interesting field of simulation modelling as carried out by experts.

The variance ratio of these methods is 3:1 in favour of (ii). (Variance reduction by a factor of 10 or more can be obtained by using stratified sampling, and by a factor of 10^4 upwards using antithetic variables).

(B) The Percolation Problem. (for example, water soaking into a sponge or brick) If the 'porosity' of the substance is known, what proportion of the interior of the material will be wetted? Tracing the motion of fluid from the outside to the interior involves impossibly large computations. It turns out that an analogous problem of tracing percolation flow outwards from a central source is (i) mathematically equivalent, (ii) much easier to simulate. Further, 'many problems can be solved in one' by producing, in the course of one simulation, results for a range of porosity values rather than a single one. In this way the problem is reduced to manageable proportions.

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